Deformation Theory. I

Maxim Kontsevich

Yan Soibelman

Current address: Institut des Hautes Études Scientifiques, Bures-sur-Yvette, F-91440, France

E-mail address: `maxim@ihes.fr`

Current address: Department of Mathematics, Kansas State University, Manhattan, KS 66506, USA

E-mail address: `soibel@math.ksu.edu`

1991 *Mathematics Subject Classification.*

THIS IS A VERY PRELIMINARY DRAFT OF THE FIRST VOLUME OF THE BOOK. IT IS INCOMPLETE AND CONTAINS MANY MISPRINTS AND, PROBABLY, MISTAKES. IT IS PLACED ON MY HOMEPAGE BECAUSE OF THE HIGH DEMAND EVEN ON SUCH AN INCOMPLETE WORK. USE IT AT YOUR OWN RISK.

Plan

4.2.7. *BRST.*

4.2.8. *DGLAs and dg-manifolds associated with deformations of local systems, complex vector bundles, complex manifolds.*

4.2.9. *dg-manifolds associated with the space of holomorphic maps.* $Maps(Y, X) = Maps(Y, \Pi T^{0,1} X)$. *The latter is a non-formal dg-manifold. But we can take a formal neighborhood of a point. Then we get the deformation theory of a morphism* $f : Y \to X$.

4.2.10. *Lie algebroids= global dg-manifolds with coordinates in degrees* 0 *and* -1. *Note: the case of coordinates in degrees* 0 *and* $+1$ *corresponds to systems of equations from 4.2.1.*

4.2.11. *Deformations of representations of the fundamental group of Kähler manifold: Goldman and Millson, Simpson.*

CHAPTER 5. Operads and algebras over operads

5.1. Generalities on operads.

5.1.1. *Polynomial functors, operads, algebras.*

5.1.2. *Examples of operads.*

5.1.3. *Colored operads.*

5.1.4. *Non-linear operads.*

5.1.5. *Pseudo-tensor categories and colored operads.*

5.2. Trees.

5.3. Resolutions of operads.

5.3.1. *Topological motivation.*

5.4. Resolutions of linear operads.

5.4.1. *Filtered resolutions of algebras.*

5.4.2. *Boardman-Vogt resolution of a linear operad.*

5.4.3. *Example: BV-resolution of an associative operad.*

5.4.4. *Cofibrant BV-resolution.*

5.5. Operads as algebras over the colored operad $\mathcal{OP}$. Free operads.

5.6. Resolutions of classical operads.

5.6.1. *Generators.*

5.6.2. *Differential.*

5.7. Deformation theory of algebras over operads.

5.7.1. *Statement of the problem and two approaches.*

5.7.2. *Deformations and free operads.*

5.7.3. *Example:* A_∞*-operad and* A_∞*-algebras.*

5.7.4. *Checking the conjecture for a free operad.*

5.7.5. *Homotopical actions of the Lie algebras of derivations.*

5.7.6. *Independence of a choice of resolution of operad.*

5.7.7. *Theorem: deformation theory of operads* $As, Lie, Comm$ *are trivial, i.e. they are rigid. Colored operad* $\mathcal{OP}$ *is rigid. The operad* $Pois$ *of Poisson algebras is not rigid.*

5.8. Koszul duality for operads and algebras.

5.9. Hochschild complex and Deligne's conjecture.

Introduction

0.1. Subject of the deformation theory can be defined as "study of moduli spaces of structures". This general definition includes large part of mathematics.

For example one can speak about the "moduli space" of objects or morphisms of a category, as long as one can give a meaning to a "variation" of either of them.

Homotopy theory is a deformation theory, since we study "variations" of topological spaces under homotopies. Moduli of smooth structures, complex structures, etc. are well-known examples of the problems of deformation theory.

In algebra one can study moduli spaces of algebraic structures: associative multiplications on a given vector space, homomorphisms between two given groups, etc.

Rephrasing the well-known quote of I. Gelfand one can say that any area of mathematics is a kind of deformation theory. In addition, a typical theory in physics depends on parameters (masses, charges, coupling constants). This leads physicists to believe that the concept of the "moduli space of theories" can be useful. For example, correlators can be computed as integrals over the moduli space of fields, dualities in the string theory can be explained in terms of special points ("cusps") of the compactified moduli spaces of certain theories, etc. In other words, deformation theory can have applications in theoretical physics.

It is quite surprising that despite the importance of the subject, there is no "general" deformation theory. At the same time it is clear that there is a need in such a theory. This feeling was supported by a number of important examples which has been worked out in algebra and geometry starting from the end of 50's. Some concepts suggested by Grothendieck, Illusie, Artin, Chevalley and others seemed to be pieces of the "general" deformation theory. Role of functorial and cohomological methods became clear. In particular, "moduli space" appeared as a functor from an appropriate category of "parameter spaces" to the category of sets. The question of whether the moduli space is "good" became a separate issue dealing with representability and smoothness of this functor.

In 1985 V. Drinfeld wrote a letter which had started to circulate under the title "Some directions of work". In the letter, among other things, he suggested to develop such a "general" deformation theory. These words mean that one should develop a language and concepts sufficient for most of the existing applications (at least in algebra and geometry). Although this goal has not been achieved yet, some progress has been made. It becomes clear that, at least in characteristic zero case, the appropriate language of the local deformation theory is the language of differential-graded manifolds (dg-manifolds for short). The latter notion is the formal version of that of a Q-manifold introduced in physics (A. Schwarz). Mathematically, a dg-manifold is a smooth scheme in the category of $\mathbf{Z}$-graded vector spaces equipped with a vector field d_X of degree $+1$, such that $[d_X, d_X] = 0$. One of the purposes of this book is to develop local deformation theory in the framework of formal pointed dg-manifolds (which are dg-manifolds with marked points).

0.2. What properties should satisfy the "moduli space" of some structures?

Let us consider an example of the moduli space of complex structures on a complex manifold. We should prove first that our moduli space is non-empty. Then we want to equip it with some additional structures, for example, a structure of an algebraic or complex variety. Then we want to study the moduli space locally and globally.

The following two questions are fundamental:

1) Is the moduli space smooth?

2) Is it compact?

First question is of local nature while the second one is global. Our point of view is that answers to the both questions are in a sense always positive. This point of view needs a justification, because a typical moduli space is a "space with singularities", not a manifold. Moduli spaces considered as sets often arise as sets of equivalence classes.

Here are few examples:

a) equivalence classes of finite sets, where equivalence is a bijection;

b) finite simple groups with the equivalence given by an isomorphism;

c) moduli space of curves of genus g (notation M_g), with an equivalence given by an algebraic automorphism. The spaces M_g are not smooth, but the non-smoothness is controlled, so one can say that "essentially" M_g is smooth;

Suppose that we have a space which is defined as a "set of equivalence classes". We need tools to prove compactness and smoothness of such a space. Tools come from algebraic geometry (geometric invariant theory for smoothness) and analysis (compactness theorems, Fredholm properties and the like). For smoothness, one also has the resolution of singularities (which changes the space), Lie groups and homogeneous spaces methods, general position arguments, Sard lemma, etc.

0.3. Let us discuss the intuitive picture of a moduli space of structures (we do not specify the type of structures).

Let V be a (possibly infinite-dimensional) vector space containing a closed subspace S of "structures given by some equations".

Example 7.0.1. Let A be a vector space, V consists of all linear maps $m : A\otimes A\to A$, and S is a subspace of such maps m that $m(m\otimes id)=m(id\otimes m)$ (as maps $A^{\otimes 3}\to A$). Then S is the space of all structures of an associative non-unital algebra on A.

Example 7.0.2. Let X be a closed smooth manifold, V be the space of almost complex structures (locally it is a vector space), S be the subspace of integrable complex structures.

Next, one has a (generally infinite-dimensional) Lie group acting on V and preserving S. In the first of the above examples it is the group of linear automorphisms $m\to m'$ of V which induce isomorphisms of algebras $(A,m)\to(A,m')$. In the second example it is the infinite-dimensional group of diffeomorphisms of X (it acts on the space of almost complex structures preserving the integrability condition).

One can define the moduli space of the structures as $\mathcal{M}=S/G$ (e.g. equivalence classes of complex structures in the second example).

Let us fix a point m in the moduli space $\mathcal{M}$ and pick a representative $\tilde{m}$ in S. Then we consider the orbit $G\tilde{m}$, which is a smooth manifold, pick a transversal manifold ("slice") T, and intersect it with S to get a space whose germ at $\tilde{m}$ is called a miniversal (or transversal) deformation.

Then one can prove the following result.

THEOREM 7.0.3. *Any family of structures containing $\tilde{m}$ is induced from the miniversal deformation. Any two miniversal deformations are isomorphic.*

Here we understand the words "family of structures" as a smooth map $\Lambda \to S$. The space Λ is called the base of family. Two families with the same base Λ are equivalent if there is a smooth family of elements $g(\lambda) \in G, \lambda \in \Lambda$ which transforms one into another one. There is an obvious notion of a pull-back of a family under a morphism of bases. Then the Theorem means that one can project an arbitrary family along the orbits of G onto a transversal slice.

If the stabilizer of $\tilde{m}$ is discrete then the miniversal deformation is the universal deformation, which means that it is unique (the equivalence between any two realizations is canonical).

0.4. Let us return to the "general" deformation theory. Suppose we have some class $\mathcal{C}$ of mathematical structures and a category of "parameter spaces" $\mathcal{W}$, such that each space $\Lambda \in Ob(\mathcal{W})$ has a "marked point" w_0. Appropriate definitions can be given in a very general framework. Suppose that we can speak about families of structures of type $\mathcal{C}$ parametrized by a parameter space Λ. Finally, suppose we can define the fiber of such a family over the marked point. Then we can state "naively" the problem of deformation of a given structure X_0 of type $\mathcal{C}$ (if $\mathcal{C}$ is a category then X_0 is an object of the category). Namely, we define the "naive" deformation functor $Def^{X_0} : \mathcal{W} \to Sets$ such that $Def^{X_0}(\Lambda)$ is the set of equivalence classes of families of objects of type $\mathcal{C}$ parametrized by Λ and such that the fiber over the marked point w_0 is equivalent to X_0 (for example isomorphic to X_0, if $\mathcal{C}$ is a category). The deformation problem is, by definition, the same thing as the functor Def^{X_0}, which is called the *deformation functor*.

According to A. Grothendieck a functor from the category $\mathcal{W}$ to the category *Sets* should be thought of as a "generalized space". One can ask whether this functor is representable. If the answer is positive, we call the representing object $\mathcal{M} \in \mathcal{W}$ the *moduli space* of the deformation problem. If the answer is negative, we still can hope that the deformation functor is ind-representable or pro-representable. For example if $\mathcal{W}$ is the category of schemes, we can hope that $\mathcal{M}$ is an ind-scheme, if it is not an ordinary scheme.

Subject of this book is the *formal* deformation theory. This means that $\mathcal{M}$ will be a "formal space " (e.g. a formal scheme). A typical category $\mathcal{W}$ will be the category of affine schemes which are spectra of local Artin algebras. Such an algebra R has the only maximal ideal m_R, which is nilpotent. Then $k = R/m_R$ and the natural embedding $Spec(k) \to Spec(R)$ defines a marked point $w_0 \in Spec(R)$. This situation was studied in the literature in 60's. A typical result says that $\mathcal{M}$ exists as a formal scheme.

Looking at the naive picture of the moduli space discussed in the previous subsection we see that the quotient space S/G can be "bad" (for example, can have singularities, even if S is smooth). This complicates the study. The idea which emerged later (P.Deligne) was to avoid factorization by the action of G. One can think of S as of *groupoid*, i.e. a category (points of S are objects,

$Hom(s_1, s_2) = \{g \in G | g(s_1) = s_2\}$) such that all morphisms are isomorphisms. Moreover, returning to the deformation functor Def^{X_0}, we see that $Def^{X_0}(\Lambda)$ is a groupoid of equivalence classes. In this way we avoid complications typical for the geometric invariant theory, i.e. we do not need to define "bad quotients". On the other hand, we are losing the notion of the moduli space, which makes the deformation theory complicated. For example: can we say when two deformation problems are equivalent, in a sense that they define isomorphic moduli spaces?

The point of view presented in this book (it goes back to the ideas of Deligne, Drinfeld, B. Feigin) is that one can overcome the difficulties at least in the case of formal deformation theory in characteristic zero case. It is based on the observation that to a formal pointed dg-manifold $\mathcal{N}$ one can assign a functor $Def_{\mathcal{N}} : Artin_k \to Sets$, where $Artin_k$ is the category of Artin local algebras over the ground field k of characteristic zero. we discuss the construction in Chapter 3. Roughly speaking $Def_{\mathcal{N}}(R)$ consists of solutions to a differential equation, modulo symmetries. An experimental fact (and, perhaps, a meta-theorem) is that for any deformation problem in characteristic zero case one can find a formal pointed dg-manifold $\mathcal{N}$ such that Def^{X_0} is isomorphic to $Def_{\mathcal{N}}$. Replacing Def^{X_0} by $Def_{\mathcal{N}}$ we simplify the deformation problem by linearizing it. In fact $\mathcal{N}$ is modeled by (generalization) of a differential-graded Lie algebra (DGLA for short). One says that this DGLA *controls* the deformation theory of X_0. All equivalence problems for the deformation theory now can be solved in terms of quasi-isomorphism classes of DGLAs. In particular, we can study the question of smoothness of the moduli space at a given point by chosing a simple representative of the quasi-isomorphism class of the DGLA controlling the deformation problem. Main theorem of the deformation theory proved in Chapter 3 asserts that the deformation functor $Def_{\mathcal{N}}$ gets replaced to isomorphic one, if we replace $\mathcal{N}$ by a quasi-isomorphic formal pointed dg-manifold (basically, it is the same as a quasi-isomorphic DGLA).

0.5. Let us say few words about the history of this book. It goes back to the lecture course given by the first author at the University of California (Berkeley) in 1994. The lectures were word processed by Alan Weinstein and later converted into TEX format. Second author used those notes in his graduate course at Kansas State University in 1997. Numerous improvements and new results were added at that time. Subsequently, we decided to write a book on deformation theory. In the course of writing we used Lectures-94 along with improvements and additions of Lectures-97. We added new results, mostly ours. We revisited the main concept of formal pointed dg-manifold from the point of view of algebraic geometry in tensor categories. We found the relationship of main structures of the deformation theory with quantum field theory. After all we realized that the project "Deformation theory" became too large to be covered in one book. We decided to split it into (at least) two volumes.

0.6. About contents of the book. Chapter 1 is devoted to the elementary examples of deformation problems. We will see how homological algebra appears as a tool for describing answers. We also discuss two important points: Schlessinger's representability theorem and Deligne's approach via DGLAs.

In Chapter 2 we review some aspects of tensor categories. We believe that many constructions of this book can be generalized to a wide class of tensor categories which are "similar" to the category of vector spaces. An example is the formal differential geometry of schemes. In Chapter 2 we briefly discuss supermanifolds.

We remark that supermanifolds equipped with the action of the group $U(1) = S^1$ are prototypes of formal pointed dg-manifolds.

Chapter 3 is devoted to systematic study of formal pointed dg-manifolds. We start with brief discussion of formal manifolds. Main idea is to consider **Z**-graded ind-schemes which correspond to cocommutative coalgebras. We call such ind-schemes *small*. Smooth small schemes correspond to cofree tensor coalgebras, i.e. as graded vector spaces they are isomorphic to $\oplus_{n\geq 1} V^{\otimes n}$ for some graded vector space V. We discuss formal differential geometry of smooth small schemes (i.e. formal pointed manifolds). In particular, we prove inverse function theorem and implicit function theorem. Vector fields on such a manifold form a graded Lie algebra. Since the manifold is **Z**-graded, we can speak about degree of a vector field. In this way we arrive to the notion of formal dg-manifold and formal pointed dg-manifold. We discuss the theory of minimal models and homotopy classification of formal pointed dg-manifolds. Tangent space to a formal pointed dg-manifold at the marked point carries a structure of complex (tangent complex). Moreover, if V is the tangent space, then we have an infinite number of polylinear maps $b_n : \bigwedge^n \to V[2-n]$, satisfying a system of quadratic relations. If $b_n = 0$ for $n \geq 3$ we get a structure of DGLA on V. In general V becomes a so-called L_∞-algebra. Finally, we explain how to associate a deformation functor with a formal pointed dg-manifold. In particular, any DGLA (or L_∞-algebra) gives rise to a deformation functor.

Chapter 4 is devoted to examples of deformation problems, both of algebraic and geometric nature. In particular, we return to the examples of Chapter 1 and discuss them in full generality, constructing formal pointed dg-manifolds, which control corresponding deformation theories.

Chapter 5 is devoted to the deformation theory of algebras over operads and PROPs. This theory covers most of algebraic examples considered previously. There are three approaches to the deformation theory of algebras over operads:

a) the "naive" one, as we discussed above;

b) the one via resolutions of algebras;

c) the one via resolution of the operad itself.

Comparing b) and c) we stress that the approach c) is more general. In particular it works in the case of PROPs as well.

There is natural resolution of any operad, so-called Boardman-Vogt resolution. We explain the construction and illustrate it in the case of associative algebras. Boardman-Vogt resolution plays an important role in the proof of Deligne conjecture about the Hochschild complex of an associative algebra. We do not discuss the proof in this book, although we state the result.

Chapter 6 is devoted to A_∞-algebras. In a sense, this chapter is a non-commutative version of Chapter 3. In particular, we have the notion of non-commutative small scheme and non-commutative formal pointed dg-manifold. Then A_∞-algebras appear in the same way as L_∞-algebras appeared in Chapter 3. In the second volume of the book we are going to discuss more general notion of A_∞-category. From this point of view, Chapter 6 can be thought of as a theory of A_∞-category with one object. Among the topics which we omitted in the last moment was the non-commutative version of Hodge-de Rham theorem (or, rather conjecture). We decided not to include it in the first volume of the book because the notion of saturated A_∞-category (non-commutative analog of a smooth projective variety) needs more explanations than we can present here. We conclude the chapter with the discussion of non-commutative volume forms and symplectic manifolds.

Appendix contains some technique and language used in the book. In particular we discuss the terminology of ind-schemes and non-commutative schemes used in Chapters 3 and 6.

About the style of the book. All new concepts (and some old ones as well) are carefully defined. At the same time, in order to save the space, we made many technical results into exercises. The reader can either accept them without proofs or (better) try to do them all. Many concepts are discussed in different parts of the book from different points of view. We believe that such a repetition will help the reader in better understanding of the topic.

Acknowldegments. We thank to Pierre Deligne for comments on the manuscript. Second author thank Clay Mathematics Institute for financial support of him as a Book Fellow. He is also grateful to IHES for excellent research conditions.

CHAPTER 1

Elementary deformation theory

1. Algebraic examples

1.1. Associative algebras. Let k be a field and A an n-dimensional associative algebra over k. For a chosen basis $(e_i)_{1\le i\le n}$ of a vector space A we define the structure constants $c_{ij}^m \in k$ as usual:

$$e_i e_j = \sum_{1\le m\le n} c_{ij}^m e_m.$$

The vector space V of structure constants has dimension n^3. Let $S \subset V$ be the subvariety of associative products on the vector space A. The associativity of the product gives rise to the following system of quadratic equations, which define S as an algebraic subvariety of V:

$$\sum_p c_{ij}^p c_{pm}^t = \sum_p c_{ip}^t c_{jm}^p$$

Here $1 \le i, j, m, t \le n$.

The group G acting on V (see Introduction) coinsides with the group $Aut(A)$ of automorphisms of A as a vector space: we can change a linear basis without changing the isomorphism class of the algebra. The "moduli space" of associative product on A is $\mathcal{M} = S/G$. Let us describe the tangent space $T_{[A]}\mathcal{M}$ at a given point $[A] = (A, (c_{ij}^m))$.

For a one-parameter first order deformation of an associative product we can write $c_{ij}^m(h) = c_{ij}^m + \tilde{c}_{ij}^m h + O(h^2)$. In order to describe the set of such deformations we impose the associativity conditions modulo h^2 and factorize by the action of the group linear transformations of the type $e_i \mapsto g_{ij} e_j$, where $g_{ij} = \delta_{ij} + h\tilde{g}_{ij} + O(h^2)$, $\tilde{g}_{ij} \in End_k(A)$.

Equivalently, we can consider all associative algebra structures on the $k[h]$-module $A_h = A[h]/(h^2)$, which extend the given one.

Let us denote by $a * b = ab + hf(a,b) + O(h^2)$ such a product. The associativity condition leads to the following equation on f:

$$f(ab, c) + f(a, b)c = f(a, bc) + af(b, c).$$

The group of symmetries consists of $k[h]$-linear automorphisms of the $k[h]$-module A_h which reduce to the identity map when $h = 0$. Such automorphisms are of the form $T(a) = a + hg(a)$, where $g : A \to A$ is an arbitrary linear map. Clearly T is invertible with the inverse given by $T(a) = a - hg(a)$.

The new product $a *' b = T(T^{-1}(a) * T^{-1}(b))$ is given by $a *' b = ab + hf^{'}(a, b)$, where

$$f^{'}(a,b) = f(a,b) + g(a)b + ag(b) - g(ab).$$

We can organize these equations into a complex of vector spaces

$$\mathrm{Hom}(A,A) \xrightarrow{d_1} \mathrm{Hom}(A \otimes A, A) \xrightarrow{d_2} \mathrm{Hom}(A \otimes A \otimes A, A),$$

where

$$d_1(g)(a,b) = g(a)b + ag(b) - g(ab),$$

$$(d_2 f)(a,b,c) = f(ab,c) + f(a,b)c - f(a,bc) - af(b,c).$$

Summarizing the above discussion, we conclude that there is a bijection
$T_{[A]}\mathcal{M} = \{\text{equivalence classes of 1st order deformations}\} \simeq \mathrm{ker} d_2/\mathrm{im} d_1$.

We can extend the above complex by adding one term to the left, $d_0 : A \to \mathrm{Hom}(A,A)$ such that $d_0(a)(x) = ax - xa$. Then the space $\mathrm{ker} d_1/\mathrm{im} d_0$ is isomorphic to the space {derivations /inner derivations}.

The above complex coincides with the first few terms of the *Hochschild complex*. Its cohomology groups are called *Hochschild cohomology* of A with the coefficients in A. We will denote them by $HH^*(A)$. We have defined only lower cohomology. The general case plays an important role in the deformation theory. We will study it later from various points of view.

Remark 1.1.1. The reader should notice that we use the same name for the extended Hochschild complex (with d_0) and for the ordinary (or truncated) Hochschild complex (without d_0). We hope this terminology will not lead to a confusion. We will make it more precise later.

First few Hochschild cohomology groups admit natural interpretation:

$$HH^0(A) = \text{center of } A,$$

$$HH^1(A) = \text{exterior derivations of } A,$$

$$HH^2(A) = \text{1st order deformations of } A.$$

Moreover, later will will define Hochschild comology of all orders. Then one will see that

$$HH^3(A) = \text{obstructions to deformations of } A.$$

More precisely, trying to extend the first order associative product to the product modulo h^3 one gets an obstruction element in $HH^3(A)$. It can be shown that if the obstruction vanishes, then every first order deformation of an associative product on A can be extended to a formal series deformation which gives an associative product modulo $O(h^n), n \geq 1$.

Exercise 1.1.2. Derive the formula for the obstruction and prove the latter statement.

What is the meaning of the higher cohomology? The following analogy was suggested by I.M.Gelfand. We know the geometric meaning of the first derivative (slope) and of the second derivative (curvature), and of the vanishing of the second derivative (inflection). The higher derivatives do not have individual meanings, but they are coefficients of the Taylor series. In the same way, one can think of all the cohomology groups as the "Taylor coefficients" of a single object. As we will see later, higher cohomology groups are encoded in the structure of the (super) moduli space of associative algebras. This super (or rather **Z**-graded) moduli space is an example of a differential-graded manifold. We will study dg-manifolds in detail in Chapter 3.

1.2. Deformations of Lie algebras. Let g be a finite-dimensional Lie algebra over the field k. We will assume that $char\, k = 0$. In order to develop the deformation theory of the Lie algebra structure on g one can proceed similarly to the case of associative algebras. The corresponding complex is called *Chevalley complex* of the Lie algebra g.

EXERCISE 1.2.1. Write down first few terms of the Chevalley complex and interpret its cohomology groups $H^*(g, g)$.

In particular, first order deformations of g are in one-to-one correspondence with the elements of the cohomology group $H^2(g, g)$. This classical result goes back to Eilenberg and MacLane.

1.3. Deformations of commutative algebras. Let us consider the deformation theory of non-unital commutative associative algebras. Again, the considerations are similar to the associative case. As a result we obtain the complex (called *Harrison complex* of a commutative algebra) which plays the same role as the Hochschild complex for associative algebras. Its cohomology $H^*(A)$ are called Harrison cohomology of the commutative algebra A.

EXERCISE 1.3.1. Write down first few terms of the Harrison complex and interpret its cohomology groups.

Not surprisingly, the second cohomology group $H^2(A)$ parametrizes the first order deformations of A. Thus we have the deformation theory which is similar to the case of associative algebras.

At this point the reader might think that the formal deformation theory we have started to discuss will be sufficient for all purposes.

We would like to warn such a reader that some interesting structures are missing in formal deformation theory.

For example, let $A = \mathbf{C}[x_1, ..., x_n]/(f_1, ..., f_m)$ where $f_i, 1 \leq i \leq m$ are some polynomials. Suppose that the algebraic variety given by the equations $f_i = 0, 1 \leq i \leq m$ is smooth. "Closed points" of this smooth affine algebraic variety are homomorphisms from A to $\mathbf{C}$.

For such varieties, the Harrison cohomology groups of the function algebra is zero in all degrees greater than 1. But the varieties are deformable in general. This means that the Harrison cohomology "feels" only the singularities. This example demonstrates limitations of the formal deformation theory. An example of a different kind is given in the next subsection.

1.4. Exercise. Let A_λ be $\mathbf{C}[x_1, x_2, x_3, x_4]$ with the relations

$$x_2 x_1 = 1,$$
$$x_3(x_1 - 1) = 1,$$
$$x_4(x_1 - \lambda) = 1.$$

1. Construct a basis $e_i(\lambda)$ of A_λ ($\lambda \in \mathbf{C}$) such that the structure constants are rational functions in λ.

2. Prove that $HH^2(A_\lambda) = 0$, i.e. the formal first order deformation theory is trivial for each value of λ.

3. Prove that A_λ and A_μ are isomorphic iff μ belongs to the set $\{\lambda, 1/\lambda, 1-\lambda, 1/(1-\lambda), \lambda/(1-\lambda), (\lambda-1)/\lambda\}$.

Thus, fixing $\lambda = \lambda_0$ appropriately, we can construct a large family of non-equivalent deformations of A_{λ_0}, which are not visible at the level of formal deformation theory.

We conclude that the formal deformation theory has limitations for infinite dimensional algebras. More sophisticated example of this kind is provided by variations of Hodge structures (or deformations of pure motives) which is not covered by the general approach advocated in this book.

Summarizing the discussion of all algebraic examples discussed above we can say that the deformation theories of associative, Lie and commutative algberas have many common features. In particular in all three cases the first order deformations are parametrized by the second cohomology of some standard complex, which can be explicitely constructed in each case. First order deformations describes the tangent space to the moduli space of structures. Hence the second cohomology of the standard complex can be thought of as the tangent space to the moduli space. We will see later that it is more natural to shift the grading so that the tangent space is given by the *first* cohomology group.

2. Geometric examples

In this section we are going to consider some geometric examples. We will see that typically the moduli space can be described (locally) in terms of the Maurer-Cartan equation.

2.1. Local systems. Let X be a topological space (say, a CW complex), G a Lie group. We denote by G^δ the group G equipped with the discrete topology. We will refer to G^δ-bundles as "local systems".

One can see three different descriptions of local systems.

A. Sheaf theoretic. A local system is given by a covering U_i of X by open sets, transition functions $\gamma_{ij} : U_i \cap U_j \to G$ which are locally constant and satisfy the 1-cocycle condition $g_{ij}g_{jk}g_{ki} = id$. Equivalence of local systems is given by a common refinement of two coverings and a family of maps to G which conjugate one system of transition functions to the other.

B. Group theoretic. Suppose that X is connected. Then equivalence classes of local systems are in one-to-one correspondence with the equivalence classes of homomorphisms of the fundamental group $\pi_1(X)$ to G. (If X is not connected, one can use the fundamental groupoid instead of the fundamental group.)

C. Differential geometric. If X is a smooth manifold, the equivalence classes of local systems on X are in one-to-one correspondence with the points of the quotient space of the space of flat connections on G-bundles modulo gauge transformations.

These three pictures give rise to three pictures of the deformation theory of local systems.

Since G is a Lie group, one can speak about local system depending smoothly on parameters, thus we have a well-defined notion of the first order deformation of a local system.

In terms of the description A, first order deformations of a local system E are in one-to-one correspondence with the equivalence classes of pairs $(\tilde{E}, i)$, where $\tilde{E}$ is a

TG-local system (TG is the total space of the tangent bundle of G), and i is an isomorphism between E and the G-local system induced from $\tilde{E}$. Let us comment on this from the algebraic point of view. Points of G are continuous homomorphisms from $C^\infty(G)$ to $\mathbf{R}$. Points of TG are continuous homomorphisms of $C^\infty(TG)$ to the ring of dual numbers $\mathbf{R}[h]/(h^2)$. This argument becomes even more transparent when G is an algebraic group. Then we can take K-points $G(K)$ for any commutative ring K. The description A gives the first order deformation theory of a local system in the form of transition functions $\gamma_{ij} : U_i \cap U_j \to G(\mathbf{R}[h]/(h^2))=$ $Hom(Spec(\mathbf{R}[h]/(h^2)), G) = TG$. The cocycle conditions give rise to a local system on TG.

EXERCISE 2.1.1. Let A be any commutative associative $\mathbf{R}$-algebra of finite dimension containing a nilpotent ideal of codimension 1. Then continuous functions from $C^\infty(G)$ to A naturally form the algebra of functions on a Lie group.

The description A gives the first order deformations as the Čech cohomology $H^1(X, \mathrm{ad}E)$, where $\mathrm{ad}E$ is the sheaf of Lie algebras associated with the principal G-bundle E.

The description B gives first order deformations of a homomorphism ρ as the first cohomology of $\pi = \pi_1(X, x)$ with coefficients in $\mathrm{ad}\rho$.

The description C gives first order deformations as the first de Rham cohomology of X with coefficients in the flat bundle $\mathrm{ad}E$.

2.2. Holomorphic vector bundles. Let X be a complex manifold. We can describe a complex structure on a smooth vector bundle $E \to X$ in two different ways.

Description A. Here we have an open cover $X = \cup_i U_i$, with holomorphic transition functions $g_{ij} : U_i \cap U_j \to GL(N, \mathbf{C})$ satisfying the 1-cocycle condition on $U_i \cap U_j \cap U_k$, namely $g_{ij}g_{jk}g_{ki} = id$

Description B. Here we have flat connections in $\bar{\partial}$-directions. Suppose that E is a smooth vector bundle over X. The complexified tangent bundle $T_X \otimes \mathbf{C}$ splits canonically into a direct sum of smooth sub-bundles $T^{1,0} \oplus T^{0,1}$ (called holomorphic and antiholomorphic sub-bundles respectively). Moreover, the antiholomorphic subbundle $T^{0,1}$ is a formally integrable distribution: if vector fields v_1, v_2 are section of $T^{0,1}$ then the Lie bracket $[v_1, v_2]$ is also a section of this bundle.

The decomposition $T_X \otimes \mathbf{C} = T^{1,0} \oplus T^{0,1}$ gives rise to the decomposition of the space of de Rham 1-forms: $\Omega^1(X) = \Omega^{1,0} \oplus \Omega^{0,1}$. A connection in the $\bar{\partial}$-direction is by definition a $\mathbf{C}$-linear map from the space of sections of E to the space of sections of $E \otimes \Omega^{0,1}$ satisfying the Leibniz formula

$$\bar{\nabla}(f\xi) = f\bar{\nabla}\xi + \xi \otimes \bar{\partial} f,$$

for an arbitrary smooth function f.

Now we can extend $\bar{\nabla}$ to a differential on $\oplus_{k\geq 0}\Gamma(X, E) \otimes \Omega^{0,k}$ (flatness guarantees that the square of this differential is zero).

THEOREM 2.2.1. *(corollary of the Newlander-Nirenberg theorem). Holomorphic structures on a smooth vector bundle are in* 1-1 *correspondence with flat $\bar{\partial}$-connections.*

So we find that the first order deformations in the picture B are given by the first Dolbeault cohomology of $H^{0,1}(X, \mathrm{End}E)$.

2.3. Deformations of complex structures. Let X be a smooth manifold equipped with a complex structure.

Description A. In this description we use the language of charts and transition functions. Thus we have homeomorphisms $f_i : U_i \to \mathbf{C}^n$ with transition functions $g_{ij} : \mathbf{C}^n \to C^n$ given by holomorphisms (i.e. isomorphisms in the category of complex manifolds) satisfying the 1-cocycle condition $g_{ij}g_{jk}g_{ik} = id$.

Description B. Here we start with a smooth manifold X with integrable almost complex structure. By definition, an almost complex structure on X is given by a decomposition $T_X \otimes \mathbf{C} = T^{1,0} \oplus T^{0,1}$ of the complexified tangent bundle. The integrability of $T^{0,1}$ is equivalent (Newlander-Nirenberg theorem) to the fact that our almost complex structure is in fact a complex one. Equivalently, one defines a $\overline{\partial}$-linear operator acting from the sheaf C_X^∞ to the sheaf $C_X^\infty \otimes (T^{0,1})^*$ (it is given by the composition of the de Rham differential with the projection to $C_X^\infty \otimes (T^{0,1})^*$). Integrability is equivalent to the condition $\overline{\partial}^2 = 0$. Deformation of a complex structure is given by a map to the tangent space of the appropriate Grassmannian. In particular, first order deformations are sections γ of the bundle $Hom(T^{0,1}, T^{1,0})$. In other words, they are "Beltrami differentials" i.e. $(0,1)$-forms with values in the holomorphic tangent bundle $T^{1,0}$. Indeed, let $\{\partial/\partial\overline{z}_j\}_{j=1}^{j=n}$ be the basis of $T^{0,1}$ corresponding to a choice of local coordinates $(z_j, \overline{z}_j), 1 \le j \le n$. It is easy to see that a "small perturbation" of this almost complex structure can be transformed by the group of diffeomorphisms $Diff(X)$ into a distribution of subspaces in the complexified tangent bundle of X spanned by $\{\partial/\partial\overline{z}_j + \sum_i \nu_{ji}(z,\overline{z})\partial/\partial z_i\}_{j=1}^{j=n}$, where $\nu_{ji}(z,\overline{z})$ are smooth functions. The formal integrability condition of this distribution becomes $\bar{\partial}\gamma = 0$. Solutions to this equation give rise to first order deformations of the complex structure. In order to obtain the set of equivalence classes of such deformations one has to factorize by the image of $\bar{\partial}$. Indeed, the deformed complex structure is given by new $\overline{\partial}$-operator of the form $\overline{\partial} + \sum_i \gamma_i \partial/\partial z_i$, where $\gamma_i = \sum_j \nu_{ji}(z,\overline{z})d\overline{z}_j$. The integrability condition for the deformed structure can be written in the form $(\overline{\partial} + \sum_i \gamma_i \partial/\partial z_i)^2 = 0$. Let us denote $\sum_i \gamma_i \partial/\partial z_i$ by γ. Then we arrive to the Maurer-Cartan equation

$$\overline{\partial}\gamma + \frac{1}{2}[\gamma,\gamma] = 0,$$

where the Lie bracket is defined naturally by means of the commutator of vector fields and the wedge product of differential forms.

In the first order deformation theory we can forget about the quadratic term $[\gamma,\gamma]$. Then we arrive to the holomorphicity condition $\overline{\partial}\gamma = 0$ for $\gamma \in \Omega^{0,1} \otimes T^{1,0}$. Clearly for a smooth function ε, the form $\gamma + \overline{\partial}\varepsilon$ defines an equivalent complex structure.

Thus one represents the tangent space to the moduli space of deformations of a given complex structure as the first Dolbeault cohomology of X with values in the holomorphic tangent sheaf $T_X^{1,0} = T^{1,0}$. The Description A gives the Čech cohomology with values in the same sheaf.

We see that in the last two examples the tangent space is of the form $H^1(X, F)$, where F is a sheaf of Lie algebras. Moreover, we have an explicit complex computing this cohomology. In all situations (algebraic and geometric), the explicit complex which computes the cohomology is a differential graded Lie algebra (DGLA for short).

Although we will discuss the approach via DGLAs Chapter 3, it might be convenient to give the definition now.

DEFINITION 2.3.1. A differential graded Lie algebra (DGLA for short) over a field k is given by the following data:

a) $\mathbf{Z}$-graded k-vector space $g = \oplus_{n\in\mathbf{Z}} g^n$;

b) brackets $(a,b) \mapsto [a,b]$ from $g^n \times g^l$ to g^{n+l};

c) linear maps d_n from g^n to g^{n+1}, such that $d = \sum_n d_n$ satisfies the condition $d^2 = 0$;

d) graded antisymmetry and graded Jacobi identity for the brackets;

e) graded derivation formula $d[a,b] = [da,b] + (-1)^{|a|}[a,db]$,

where a, b are homogeneous elements of g of degrees $|a|, |b|$ respectively.

General idea is that (locally) any kind of moduli space can be described as a quotient of the set of elements $\gamma \in g^1$ satisfying the Maurer-Cartan equation $d\gamma + \frac{1}{2}[\gamma,\gamma] = 0$ by the action of the group corresponding to the Lie algebra g^0. All algebraic and geometric examples discussed above (as well as many other which we will discuss later) are special cases of this approach.

3. Schlessinger's axioms

In this section we are going to sketch an approach to the deformation theory which goes back to Grothendieck and was further developed in the paper by Michael Schlessinger "Functors of Artin rings", Transactions of AMS, 130:2 (1968), 208-222. Notice that in this approach we are not required to work over a field of characteristic zero.

DEFINITION 3.0.2. A commutative unital associative ring A is called Artin if every descending chain of ideals in A stabilizes.

Then one has the following result (the proof is left to the reader)

PROPOSITION 3.0.3. *a) An Artin ring A is a finite direct sum of Artin rings $A = \oplus_i A_i$ such that:*

a) every A_i is a local ring;

b) the maximal ideal $m_i \subset A_i$ is nilpotent;

c) for any $N \geq 1$ the quotient space A_i/m_i^N is a finite dimensional vector space over the field $k_i = A_i/m_i$.

Let us fix an arbitrary ground field k and denote by $Artin_k$ the category of Artin local k-algebras. Objects of $Artin_k$ are, by definition, Artin local k-algebras A such that

a) $A/m \simeq k$ where $m = m_A$ is the maximal ideal of A;

b) $A \simeq k \oplus m$ as a k-vector space;

c) the ideal m is nilpotent.

Morphisms in the category $Artin_k$ are homomorphisms of unital k-algebras.

EXAMPLE 3.0.4. Let $A = k[h]/(h^n), n \geq 1$. Then A is an Artin local k-algebra with the maximal ideal (h). Similarly $A = k[h_1, ..., h_l]/(h_1, ..., h_l)^n$ is the local Artin k-algebra with the maximal ideal $(h_1, ..., h_l)$.

In general, if we deform a mathematical structure X_0, we have a family of structures X_t parametrized by a parameter t, so that $X_{t=0} \simeq X_0$. In fact, if we consider the formal picture ("formal moduli space") then t is a formal parameter. For example one can have $t \in k[[h]]$. If we are interested in the local structure of the formal "moduli space of structures" at the point corresponding to X_0, then we can approximate it by a sequence of "jets" $Hom_{Alg_k}(R, A_n)$, where $A_n = k[[h]]/(h^n)$ and R is the completion of the local ring of the moduli space at X_0. This sequence should be compatible with the natural homomorphisms $A_{n+1} \to A_n$. The projective system $(A_n)_{n \geq 1}$ gives rise to a functor $F : Artin_k \to Sets$ such that $F(B) = \varprojlim_n Hom_{Alg_k}(B, A_n)$. Summarizaing, we can say that the local structure of the moduli space at X_0 is completely described by the functor $F : Artin_k \to Sets$. Moreover, there exists a complete Noetherian local k-algebra R such that we have a functorial isomorphism

$$F(A) \simeq Hom_{Alg_{k,top}}(R, A),$$

where in the RHS we take all topological homomorphisms of algebras $R \to A$.

This observation suggests the following strategy. First one defines the functor of "deformations of X_0 parametrized by a Artin local k-algebra A". Then one tries to prove that that it is pro-representable, i.e. there exists a complete local k-algebra R such that the above isomorphism holds. Then the algebra R is the completion of the local ring of the moduli space at the point $[X_0]$.

In the case of schemes Schlessinger suggested a list of properties (axioms) which one should check in order to prove that the formal moduli space of deformations exists.

Let X be a scheme over the field k. Then the deformation functor D in the sense of Schlessinger assigns to an Artin local k-algebra A the isomorphism class of pairs (Y, i) such that Y is a flat scheme over $Spec(A)$ and $i : X \to Y$ is a closed immersion of X as a fiber over $Spec(k)$ (to say it differently, $X \simeq Y \times_{Spec(A)} Spec(k)$). It is clear how to define D on morphisms.

In general, this functor is not pro-representable. Nevertheless it often satisfies certain properties which "almost" imply pro-representability (at least they imply existence of R).

Here is the list of properties:

(Sch1) Let $0 \to (x) \to B \to A \to 0$ be an exact sequence such that (B, m_B) and (A, m_A) are Artin local rings, and (x) is a prinicipal ideal such that $m_B x = 0$. Then for any morphism $(B', m_{B'}) \to (A, m_A)$ the natural map $D(B \times_A B') \to D(B) \times_{D(A)} D(B')$ is surjective.

(Sch2) The above-mentioned surjection is a bijection when $A = k, B' = k[\epsilon]/(\epsilon^2)$.

(Sch3) $dim_k D(k[\epsilon]/(\epsilon^2)) < \infty$.

In algebraic geometry the tangent space to a scheme X at a smooth point $x \in X$ is given by the set of morphisms $Spec(k[\epsilon]/(\epsilon^2)) \to X$ such that $Spec(k)$ is mapped into x. Indeed, homomorphisms of k-algebras $R \to k[\epsilon]/(\epsilon^2)$ correspond to derivatives of R, hence vector fields on $Spec(R)$. More formally, let us fix a complete Noetherian local k-algebra R and consider a functor $h_R : Artin_k \to Sets$ such that $h_R(A) = Hom_{Alg_k}(R, A)$. Then $h_R(k[\epsilon]/(\epsilon^2))$ is naturally isomorphic to $Hom_{Alg_k}(R, k[\epsilon]/(\epsilon^2))$. For this reason the set $D(k[\epsilon]/(\epsilon^2))$ is called the *tangent space* to the functor D. Then the property (Sch3) says that the tangent space to D is finite-dimensional.

If a functor D satisfies the properties (Sch1)-(Sch3) then it is not necessarily pro-representable. But it is pro-representable, if an addition one has the following property

(Sch4) Let $B \to A$ be as in (Sch1). Then the natural map $D(B \times_A B) \to D(B) \times_{D(A)} D(B)$ is a bijection.

The properties (Sch1)-(Sch3) hold for the deformation functor D associated with a scheme X, which is proper over k. Some additional restrictions on X force D to be pro-representable. One can check that this approach agrees with the geometric intuition. For example, the tangent space to the deformation functor associated with a smooth scheme is naturally isomorphic to $H^1(X, T_X)$, where T_X is the tangent sheaf of X.

This approach works in the examples of Sections 1.1 and 1.2 as well. According to Grothendieck any functor from the category of schemes to the category *Sets* should be thought of as "generalized scheme". Formal schemes, as functors, are non-trivial on finite-dimensional algebras only. Suppose we want to describe the formal deformation theory of a certain mathematical structure X_0 defined over a field k (algebras of any sort, complex structures, flat bundles, etc.). This means that we classify flat families $X_s, s \in Spec(R)$, where R is a local Artin algebra, such that fiber over $s_0 = Spec(k)$ is isomorphic to X_0. This gives rise to a "naive" deformation functor $Def^{X_0} : Artin_k \to Sets$. The question is when Def^{X_0} is represented by a pro-object in the category $Artin_k$ (i.e. when it give rise to a formal scheme with marked point). We will see that this is the case in all examples from Sections 1.1 and 1.2. In the next section we will see that there is a general way to construct functors from Artin algebras to *Sets*, starting with differential-graded Lie algebras. Later we will describe differential-graded Lie algebras for all examples of Sections 1.1 and 1.2.

4. DGLAs and Deligne's groupoids

Naive approach to the deformation theory discussed in the Introduction describes the moduli space of some structures as a quotient space S/G. The latter space can be singular even if S is smooth. One should take care about "bad action" of G on S. This is the subject of the Geometric Invariant Theory. Alternatively, one can think of S/G as of groupoid. In other words, one does not factorize by the action of G but remembers that certain points of S are equivalent.

DEFINITION 4.0.5. Groupoid is a category such that every morphism in it is an isomorphism.

For example, a group G gives rise to a groupoid with one object e such that $Hom(e, e) = G$. This example can be generalized.

EXAMPLE 4.0.6. If S is a set and a group G acts on S, one can define an "action" groupoid in the natural way: objects are points of S, and $Hom(x, y)$ consists of all $g \in G$ such that $gx = y$.

Pierre Deligne suggested in 80's the following approach to the deformation theory in characteristic zero case.

One considers a category of all possible deformations of a given structure. Morphisms between objects are equivalences of deformations. Then one has a groupoid $\mathcal{S}$ "controlling" the deformation problem. Two deformation problems are equivalent if the corresponding groupoids are equivalent (as categories). The "naive" moduli space of deformations is the set of isomorphism classes $Iso(\mathcal{S})$. This space can be singular, but the idea is that all the information about the space is encoded in the groupoid $\mathcal{S}$. One can go one step further and consider sheaves of groupoids (an important special case of the latter is called *gerbe*).

The question is how to use this general approach in practice. At the end of 80's Deligne, Vladimir Drinfeld and Boris Feigin suggested that for a given deformation problem one can find a DGLA which "controlls" it. Forgetting about groupoid structure this means that the "naive" deformation functor Def^{X_0} (see previous section) is isomorphic to another one, constructed canonically from some DGLA (depending on X_0). It turns out that the approach via DGLAs automatically gives groupoids. Deligne's groupoid can be described explicitely in terms of the DGLA. It will be explained in geometric terms in Chapter 3.

Let us start with a DGLA $g = \oplus_{n\geq 0} g^n$ over a field k of characteristic zero. (i.e. it is $\mathbf{Z}$-graded without negative degree components).

Let $V = g^1$, and S be the subset of V consisting of elements γ satisfying the equation $d\gamma + \frac{1}{2}[\gamma,\gamma] = 0$.

Instead of a group G acting on S, we have the Lie algebra $g = g^0$ acting on g^1 by affine vector fields. Namely, $\alpha \in g^0$ corresponds to the following affine vector field on g^1

$$\dot{\gamma} = [\alpha,\gamma] - d\alpha.$$

PROPOSITION 4.0.7. *In this way we obtain a Lie algebra homomorphism $g^0 \to Vect(g^1)$ such that the image of g^0 preserves the equation for S.*

Proof. We will check the last condition. The first one is left to the reader as an exercise. Let $K(\gamma) = d\gamma + \frac{1}{2}[\gamma,\gamma] = 0$. Then we would like to show that $\dot{K}(\gamma) = 0$ for every α.

We use the chain rule: $\dot{K}(\gamma) = d\dot{\gamma} + [\dot{\gamma},\gamma] = d([\alpha,\gamma] - d\alpha) + [[\alpha,\gamma] - d\alpha,\gamma] = [d\alpha,\gamma] + [\alpha, d\gamma] - dd\alpha + ... = [\alpha, d\gamma] + [[\alpha,\gamma],\gamma] = -\frac{1}{2}[\alpha,[\gamma,\gamma]] + \frac{1}{2}[[\alpha,\gamma],\gamma] + \frac{1}{2}[[\alpha,\gamma],\gamma] = 0$. We used here the curvature zero condition for γ plus the Jacobi identity. ■

We would like to have a groupoid, but we do not have a group. We have a Lie algebra g^0, but the notion of the orbit space for infinite-dimensional Lie algebras is complicated. In order to overcome the difficulty we are going to use local Artin k-algebras. Namely, to a DGLA g we can associate a functor Def_g from local Artin k-algebras to groupoids.

The objects of the groupoid corresponding to a local Artin k-algebra A with the maximal ideal m are elements $\gamma \in g^1 \otimes m$ satisfying the Maurer-Cartan equation $d\gamma + \frac{1}{2}[\gamma,\gamma] = 0$.

In order to describe morphisms, we consider the nilpotent Lie algebra $g^0 \otimes m$. To every nilpotent Lie algebra g over k we can associate the group of formal symbols $\exp(x), x \in g$, with multiplication given by the Campbell-Baker-Hausdorff formula. Proof of the following Proposition is left to the reader.

PROPOSITION 4.0.8. *The group $exp(g^0 \otimes m)$ acts on the set of objects of our category. Namely, an element ϕ of the group acts by the formula*

$$\gamma \mapsto \phi\gamma\phi^{-1} - (d\phi)\phi^{-1}$$

(compare with the action of gauge transformations on connections).

Here we use the notation

$$\exp(\alpha)\gamma\exp(-\alpha) = \sum_{n\geq 0}(\mathrm{ad}\alpha)^n(\gamma)/n!$$

Also,

$$(d\phi)\phi^{-1} = (d\exp\alpha)\exp(-\alpha)$$

is defined by

$$(d\phi)\phi^{-1} = \sum_{n\geq 0}(1/(n+1)!)(\mathrm{ad}\alpha)^n(d\alpha).$$

For $\phi = e^{t\alpha}$ we obtain

$$\gamma \mapsto e^{t\mathrm{ad}\alpha}(\gamma) + \frac{(Id - e^{t\mathrm{ad}\alpha})}{\mathrm{ad}\alpha}(d\alpha)$$

One can generalize the action to the case of finite characteristic. In order to do that one has to use divided powers in the above definitions. We are not going to do that since we are interested in the case $char(k) = 0$.

The above formula is very transparent in the case of real numbers. Indeed if $A : V \to V$ is a linear endomorphism of a vector space V, $b \in V$ then the differential equation $\dot{X} = AX + b$ with the initial condition $X(0) = X_0$ has a solution $X(t) = A^{-1}(e^{tA}(AX_0 + b) - b)$. If we understand both sides as formal power series in t, then the formula makes sense over a field of characteristic zero and any A.

Now we define a groupoid as the action groupoid of this action. In other words, $Hom(\gamma_1, \gamma_2) = \{\phi | \phi(\gamma_1) = \gamma_2\}$. The composition of morphisms is given by the group product.

Remark 4.0.9. As we will see later, sometimes it is convenient to consider graded nilpotent commutative algebras without the unit instead of Artin local algebras.

The reader has noticed that we have constructed not just a groupoid, but a functor from the category of Artin local k-algebras to the 2-category of groupoids (the notion of such a functor can be made precise, but we don't need it in this book). We say that two deformation problems are equivalent if the corresponding functors are isomorphic. In this book we do not stress the groupoid structure, thinking about Def_g as a functor from $Artin_k$ to $Sets$.

CHAPTER 2

Tensor categories

1. Language of linear algebra

Traditionally mathematical structures are defined as collections of *sets* together with certain relations between them (see [Bou]). In many cases there is an alternative way to define structures using as the basic building blocks *vector spaces* over some base field k. In this book we assume that the characteristic of k is zero.

In analytic questions field k is usually $\mathbf{R}$ or $\mathbf{C}$, and one has to put some topology on infinite-dimensional vector spaces over k. In algebraic geometry k could be arbitrary, or a non-archimedian topological field in analysis.

Here we will show several classical examples of the use of the language of linear algebra instead of set theory.

1.1. From spaces to algebras. Sets in general should be replaced (as far as possible) by "spaces" (topological spaces, manifolds, algebraic varieties, ...). As for the notion of a "space", one of ways to encode it is via the correspondence

$$\text{Space } X \leftrightarrow \text{commutative associative unital algebra } \mathcal{O}(X)$$

Often, it is not enough to have just one algebra, but one need a *sheaf* of algebras $\mathcal{O}_X$ on the "underlying topological space" of X.

1.1.1. *Smooth and real-analytic manifolds.* Any C^∞-manifold X can be encoded by the topological algebra $\mathcal{O}(X) := C^\infty(X)$ over $k = \mathbf{R}$. Analogously, a real-analytic manifold X is encoded by the algebra of real-analytic functions $C^\omega(X)$.

One can complexify these algebras. For real-analytic X we consider $C^\omega(X) \otimes \mathbf{C}$ as the algebra of functions on a "degenerate complex manifold", the germ of $X_\mathbf{C}$ of the complexification of X. Analogusly, for a smooth manifold X the algebra $C^\infty(X) \otimes \mathbf{C}$ could be viewed as the algebra of holomorphic functions on a "formal neighborhood" $X_\mathbf{C}^{formal}$ of X in non-existing complexification $X_\mathbf{C}$.

1.1.2. *Complex-analytic spaces.* Obviously a complex-analytic space X gives a sheaf $\mathcal{O}_X$ of algebras over $\mathbf{C}$ on the underlying topological space (which is usually denoted again by X). A Stein space X can be reconstructed just from one (topological) algebra $\mathcal{O}(X) := \mathcal{O}_X(X)$ (topology is defined by the uniform convergence on compact subsets).

1.1.3. *Schemes.* The category of *affine schemes* over $\mathrm{Spec}(k)$ is opposite to the category of unital commutative associative k-algebras.

In general, a scheme X over $\mathrm{Spec}(k)$ is by definition a topological space endowed with the sheaf of k-algebras $\mathcal{O}_X$ which is locally isomorphic to the standard structure sheaf on the spectrum $\mathrm{Spec}(A)$ of a k-algebra A.

In essentially all applications one uses *separated* schemes (analogous to Hausdorff spaces in the usual topology). A separated scheme X is automatically *quasi-separated*, which means that the intersection of any two open affine subschemes in X is a union of finitely many affine open sets. One can show that if $(U_i)_{i\in I}$ is an affine covering of a quasi-separated scheme X, then the pair $(X, (U_i)_{i\in I})$ is completely encoded by the following:

Data: • a set I,
- a k-algebra A_S for every finite nonempty subset $S \subset I$,
- a morphism $i_{S_1,S_2} : A_{S_1} \longrightarrow A_{S_2}$ for $S_1 \subset S_2$

Axioms: • $i_{S,S} = id_{A_S}, i_{S_1,S_3} = i_{S_2,S_3} \circ i_{S_1,S_2}$ (i.e. we get a functor),

- if $S_1 \cap S_2 \neq \emptyset$ then $A_{S_1\cap S_2}$ is equal to the tensor product $A_{S_1} \bigotimes_{A_{S_1\cup S_2}} A_{S_2}$, i.e. the functor $S \mapsto A_S$ from the poset of finite nonempty subsets in I to the cateory of algebras preserves cartesian coproducts,
- homomorphisms i_{S_1,S_2} are localizations

For the covering $(U_i)_{i\in I}$ the correpsonding algebras A_S are defined as $\mathcal{O}(\cup_{i\in S} U_i)$, and i_{S_1,S_2} as restriction morphisms. Conversely, any diagram of algebras (A_S) satisfying axioms as above gives an affine covering of a quasi-separated scheme. Thus, we can hide the theory of prime ideals and Zariski topology, and give an "elementary" definition of a (quasi-separated) scheme. In fact, it is enough to consider only subsets $S \subset I$ with at most 3 elements.

1.1.4. *A bad example: Topological spaces.* Classical theorem of I. M. Gelfand says that compact Hausdorff topological spaces are in one-to-one correspondence with commutative unital C^*-algebras over $k = \mathbf{C}$ (more precisely, we have an anti-equivalence of categories). Although this fact was extremely influential in the history of the algebraization of spaces, for purposes of deformation theory algebras of the type $C(X)$ are not good (for example, they do not admit derivations). From our point view, topological spaces are better described in the classical set-theoretic way.

1.1.5. *Dictionary between geometry and algebra.* Here are some standard correspondences (in the "affine case"):

Maps $f : X \longrightarrow Y$	morphisms of algebras $f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$
points of X (or k-points in the case of affine schemes)	$\mathrm{Hom}_{k-alg}(\mathcal{O}(X), k)$
closed embedding $i : X \hookrightarrow Y$	epimorphism of algebras $i^* : \mathcal{O}(Y) \twoheadrightarrow \mathcal{O}(X)$, equivalently, an ideal $I_Y \subset \mathcal{O}(Y)$ and an isomorphism $\mathcal{O}(X) \simeq \mathcal{O}(Y)/I_Y$
finite product $\prod_{i\in I} X_i$	tensor product $\bigotimes_{i\in I} \mathcal{O}(X_i)$ (completed if we are not in purely algebraic situation)
finite disjoint union $\coprod_{i\in I} X_i$	direct sum $\bigoplus_{i\in I} \mathcal{O}(X_i)$
vector bundle E on X	finitely generated projective $\mathcal{O}(X)$-module $\Gamma(E)$

1.2. Linearization of differential geometry. Here are translations of some notions of differential geometry (again in the "affine case"):

X is smooth	$A = \mathcal{O}(X)$ is formally smooth, i.e. for any nilpotent extension of algebras $f : B \twoheadrightarrow B/I$, with $I^n = 0$ for $n \gg 0$ the induced map $f^* : \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, B/I)$ is surjective
total space *tot* E of the vector bundle E on X, considered as manifold in the category of affine schemes	symmetric algebra generated by the $\mathcal{O}(X)$-module $M := \Gamma(E^*) = \mathrm{Hom}_{\mathcal{O}(X)-mod}(\Gamma(E), \mathcal{O}(X))$ $Sym_{\mathcal{O}(X)-mod}(M) = \bigoplus_{n\geq 0} Sym^n_{\mathcal{O}(X)-mod}(M)$
a vector field on X	a derivation of $\mathcal{O}(X)$
tangent bundle T_X	space of derivations $\mathrm{Der}(\mathcal{O}(X))$ considered as $\mathcal{O}(X)$-module
cotangent bundle T^*_X	an $\mathcal{O}(X)$-module $\Omega^1(\mathcal{O}(X)) := \mathrm{Hom}_{\mathcal{O}(X)-mod}(\mathrm{Der}\,\mathcal{O}(X)$
differential forms $\Omega^\bullet_X$	$\wedge^\bullet_{\mathcal{O}(X)-mod}(\Omega^1(\mathcal{O}(X)))$
Lie group (or an affine algebraic group)	commutative Hopf algebra

1.2.1. *Notations in differential geometry.* If E is a vector bundle on X, we consider E also as a sheaf of $\mathcal{O}_X$-modules.

Thus, $\Gamma(E) = E(X)$ is the space of sections of E.

The tangent and cotangent bundle are denoted by T_X and T^*_X respectively. The space $\Gamma(T_X) = T_X(X)$ is also denoted by $Vect(X)$. Thus, $Vect(X) = \mathrm{Der}(\mathcal{O}(X))$. Spaces $\Gamma(\wedge^k T^*_X)$ are denoted by $\Omega^k(X)$.

For vector bundle E on manifold X we denote by *tot* E the total space of E considered as a *manifold*. Again, for the special case $E = T_X$ (or $E = T^*_X$) we denote *tot* E simply by TX (or by T^*X).

1.3. Less trivial example: space of maps. We assume that we are working in some category of spaces, with sets of morphisms denoted by $Hom(X, Y)$. In many cases for two spaces X and Y one can define a new space $Maps(X, Y)$ (called inner Hom) by the usual categorical property:

$$Hom(Z, Maps(X, Y)) = Hom(Z \times X, Y)) \text{ as functors in } Z \qquad (2.1)$$

The most clean situation appears in the category of affine schemes over k.

THEOREM 1.3.1. *Let X be a finite scheme, i.e. $\mathcal{O}(X)$ is a finite-dimensional algebra, and Y be an arbitrary affine scheme. Then there exists an affine scheme $Maps(X, Y)$ satisfying the property (2.1) .*

Proof. Denote algebras of functions $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ simply by A and B. The algebra $C := \mathcal{O}(Maps(X, Y))$ should be such that we have an isomorphism functorial in R ($R = \mathcal{O}(Z)$ in the notation of (2.1))

$$\mathrm{Hom}(C, R) \simeq \mathrm{Hom}(A, R \otimes B) \tag{2.2}$$

Let $\{b_i\}$ be a basis of of k-vector space B with $b_0 = 1$. Then a homomorphism from A to $R \otimes B$ is of the form $a \mapsto \sum f_i(a) \otimes b_i$ where $f_i : A \longrightarrow R$ are linear maps. Since $1_A \mapsto 1_{R \otimes B}$, we have

$$f_0(1_A) = 1_R, f_i(1_A) = 0 \text{ for } i \neq 0 \tag{2.3}$$

The multiplicativity of the homomorphisms gives:

$$\sum_i f_i(a_1 a_2) \otimes b_i = \sum_{j,k} f_j(a_1) f_k(a_2) \otimes b_j b_k$$

If the structure constants of B are given by $b_j b_k = \sum_i c_{ijk} b_i$ we find the relations

$$f_i(a_1 a_2) = \sum_{j,k} f_j(a_1) f_k(a_2) c_{ijk}. \tag{2.4}$$

Now it is clear that if we define an algebra C generated by symbols $f_i(a)$ satisfying the relations (2.3) and (2.4), together with the relations

$$f_i(\lambda a_1 + \mu a_2) = \lambda f_i(a_1) + \mu f_i(a_2) \text{ for } \lambda, \mu \in k \tag{2.5}$$

then the functorial property 2.1 holds.■

Example 1.3.2. If $X = \mathrm{Spec}(k)$ is a point then $Maps(X, Y) = Y$. If $X = \mathrm{Spec}(k[t]/(t^2))$ and Y is smooth then $Maps(X, Y)$ coincides with TY, the total space of the tangent bundle to Y.

Exercise 1.3.3. Let us work in the category of associative unital k-algebras, not necessarily commuttative. Prove the following version of the theorem 1.3.1: for a finite-dimensional algebra B and an arbitrary algebra A there exists a canonical algebra C such that (2.2) holds. What happens if $B = k[t]/(t^2)$?

1.4. Avoiding individual vectors. In essentially all definitions of mathematical structures given in terms of linear algebra, one can restrict oneself only to formulas containing symbols Hom, Ker, $\oplus$, $\otimes$, and so on. Also, it is convenient to denote by $\mathbf{1}$ the standard 1-dimensional vector space $k^1 = k$. Elements of vector space V are the same as morphisms $\mathbf{1} \longrightarrow V$.

For example, an *associative algebra* is a pair $A = (V, m)$ where V is a vector space and $m \in \mathrm{hom}(V \otimes V, V)$ is a map such that

$$m \circ (m \otimes id_V) = m \circ (id_V \otimes m) \in Hom(V \otimes V \otimes V, V)$$

A *unital associative algebra* is a triple (V, m, u) where V and m are as before, and u is not simply an element of V, but a map $u \in \mathrm{Hom}(\mathbf{1}, V)$ satisfying additional axioms: both compositions

$$V \simeq V \otimes \mathbf{1} \xrightarrow{m} V, V \simeq \mathbf{1} \otimes V \xrightarrow{m} V$$

coincide with the identity morphism id_V, where isomorphisms $V \simeq V \otimes \mathbf{1} \simeq \mathbf{1} \otimes V$ are standard "identity" isomorphisms.

A *commutative unital associative algebra* is a triple (V, m, u) as before such that

$$m = P_{(21)} \circ m$$

where $P_{(21)} : V \otimes V \longrightarrow V \otimes V$ is the permutation $u \otimes v \mapsto v \otimes u$. In general, for any finite $n \geq 0$ and a permutation $\sigma \in \Sigma_n$ we denote by P_σ the corresponding operator on $V^{\otimes n}$:

$$P_\sigma(v_1 \otimes \cdots \otimes v_n) := v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

EXERCISE 1.4.1. Rewrite the description of the algebra C in the proof of theorem 1.3.1 without using basis of B.

1.5. Graphs and acyclic tensor calculus. Here we introduce a class of graphs which will be used widely in the book. These graphs depict ways to "contract incdices" (or "compose") several tensors.

DEFINITION 1.5.1. An *oriented graph* Γ consists of the following data:

- finite set $V(\Gamma)$, its elements are *vertices* of Γ
- finite set $E(\Gamma)$, elements are *edges* of Γ,
- two maps $head, tail : E(\Gamma) \longrightarrow V(\Gamma)$,
- decomposition of $V(\Gamma)$ into a disjoint union of three sets

$$V(\Gamma) = V_{in}(\Gamma) \sqcup V_{internal}(\Gamma) \sqcup V_{out}(\Gamma)$$

satisfying the axioms:

- for any edge $e \in E(\Gamma)$ we have

$$head(e) \notin V_{out}, tail(e) \notin V_{in}$$

- for any $v \in V_{in}(\Gamma)$ there exists unique $e \in E(\Gamma)$ such that $head(e) = v$,
- for any $v \in V_{out}(\Gamma)$ there exists unique $e \in E(\Gamma)$ such that $tail(e) = v$.

REMARK 1.5.2. We think of an edge e as being oriented from the $head(e)$ to the $tail(e)$.

For any oriented graph Γ we denote by E_{in} the set of edges with the head in V_{in}, and by E_{out} the set of edges with the tail in V_{out}. Edges which do not belong to $E_{in} \sqcup E_{out}$ are called *internal.* Notice that there could be edges which belong to both sets E_{in} and E_{out} simultaneously. Such an edge starts at a vertex in V_{in} and end at vertex in V_{out}.

For vertex $v \in V(\Gamma)$ we denote by $Star_{in}$ the set of edges $e \in E(\Gamma)$ such that $tail(e) = v$. We say that such e *ends* at v. Analogously, by $Star_{out}(v)$ we denote the set of edges e such that $head(e) = v$, and we say that e *ends* at v.

Suppose that with every edge e of oriented graph Γ we associate a finite-dimensional vector space U_e, and with every internal vertex $v \in V_{internal}(\Gamma)$ we associate a linear map

$$T_v \in \mathrm{Hom}(\otimes_{e \in Star_{in}(v)} U_e, \otimes_{e \in Star_{out}(v)} U_e)$$

Then one can define a "composition" of tensors T_v given by Γ:

$$comp_\Gamma((T_v)_{v \in V_{internal}(\Gamma)}) \in Hom(\otimes_{e \in E_{in}(\Gamma)} U_e, \otimes_{e \in E_{in}(\Gamma)} U_e)$$

The defintion is obvious: maps T_v can be considered as elements of tensor products,

$$T_v \in (\otimes_{e \in Star_{in}(v)} U_e^*) \otimes (\otimes_{e \in Star_{out}(v)} U_e)$$

Graph Γ defines a way to contract some indices in $\otimes_{v \in V_{internal}(G)} T_v$ and get the result.

For infinite-dimensional spaces linear maps can not be always identified with elements of tensor products. To compose polylinear maps in arbitrary dimensions one has to reduce the class of graphs under consideration:

DEFINITION 1.5.3. An oriented graph Γ is *acyclic* if there is no cyclic sequence of edges $(e_i)_{i \in \mathbf{Z/nZ}}, n \geq 1$, such that $tail(e_i) = head(e_{i+1})$ for all $i \in \mathbf{Z/nZ}$.

It is obvious how to define composition given by an acyclic graph for polylinear maps of *arbitrary* vector spaces. One should just choose a grading $gr : V(\Gamma) \to \mathbf{Z}$ on the set of vertices in such a way that for any $e \in E(\Gamma)$ we have $gr(head(e)) < gr(tail(e))$.

Trees form a particualr class of acyclic oriented graphs:

DEFINITION 1.5.4. A *tree* is an oriented graph T such that $V_{out(T)}$ consists of one element, which is denoted by $root_T$, and such that for any $v \in V(T), v \neq root_T$ there exists unique path $(e_1, \dots, e_n), n \geq 1$ from v to $root_T$:

$$head(e_1) = v, tail(e_1) = head(e_2), \dots, tail(e_n) = root_T$$

1.6. Why it is useful to speak algebraically? There are two basic reasons:

- Structures defined linear algebraically can be transformed to a larger realm of tensor categories (see the next section),
- Deformation theory is naturally defined for things described in terms of linear algebra.

2. Definition of a tensor category

We are going to give a definition of tensor categories, and of various facultative properties of them. Our terminology is slightly different from the standard one. In [Deligne]??? the name "tensor catgeories" is used for what we will call *rigid* tensor categories (see defintion 2.1.5). In what follows it is convenient to have in mind that the typical example of a tensor category is the category $Repr_{k,G}$ of k-linear representation of a given abstract group G (which is the same as the category $Vect_k$ of vector spaces over k if $G = \{id\}$). Morally, one can think about tensor categories as about "categories of representations without the group".

DEFINITION 2.0.1. A k-linear tensor category is given by the following data:

(1) a k-linear category $\mathcal{C}$ (which means that all morphism spaces are k-vector spaces, and compositions are bilinear),
(2) a bilinear bi-functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$,
(3) an object $\mathbf{1}_{\mathcal{C}}$ (denoted often simply by $\mathbf{1}$),
(4) a functorial in $V_1, V_2, V_3 \in Ob(\mathcal{C})$ isomorphism

$$a = assoc : V_1 \otimes (V_2 \otimes V_3) \simeq (V_1 \otimes V_2) \otimes V_3)$$

(5) a functorial in $V_1, V_2 \in Ob(\mathcal{C})$ isomorphism

$$c = comm : V_1 \otimes V_2 \simeq V_2 \otimes V_1$$

(6) an functorial in $V \in Ob(\mathcal{C})$ isomorphism

$$u = un : V \otimes \mathbf{1} \simeq V$$

satisfying the following axioms:

(1) category $\mathcal{C}$ is abelian, i.e. it contains finite direct sums, kernels and cokernels of morphisms, and the coimages are isomorphic to images,
(2) data 2.-6. give a structure of a symmetric monoidal category on $\mathcal{C}$ (see Appendix to this Chapter,
(3) morphisms of functors in data 4,5,6 are k-linear on morphisms of objects,
(4) $\mathrm{Hom}(\mathbf{1},\mathbf{1}) = k \cdot id_{\mathbf{1}}$
(5) for any object $V \in Ob(\mathcal{C})$ the functor $V \otimes \bullet : \mathcal{C} \longrightarrow \mathcal{C}$ is exact

We will describe in details the notion of a symmetric monoidal category (i.e. data 2.-6. and axiom 2 from above) in the Appendix 9. Morally, these axiom mean that for any finite collection of objects $(V_i)_{i\in I}$ one can make functorially the tensor product $\otimes_{i\in I} V_i$, which is isomorphic to

$$V_1 \otimes \cdots \otimes V_n := V_1 \otimes (V_2 \otimes \cdots \otimes (V_{n-1} \otimes V_n) \dots)$$

if $I = \{1, \dots, n\}$. In particular, on

$$V^{\otimes n} := V^{\otimes\{1,\dots,n\}} = V \otimes \cdots \otimes V (n \text{ times})$$

acts the symmetric group Σ_n. The action of $\sigma \in \Sigma_n$ we will denote by P_σ.

Notice that the *data* 2.-4. in the definition of a tensor category are completely parallel to the *axioms* in the definition of a unital commutative associative algebra (see 1.4).

REMARK 2.0.2. It is clear from the definitions that one can perform acyclic tensor calculus in an arbitrary tensor category. In particular, if Γ is a graph, and we are given a map $E(\Gamma) \to Ob(\mathcal{C})$, then the composition maps described in the Section 1.5 can be defined by the same formulas.

Finally, we would like to mention that one can drop the requirement for $\mathcal{C}$ to be k-linear. In that case we arrive to the notion of *symmetric monoidal category* considered in the Appendix. Most of the properties and constructions of this Chapter can be generalized to the case of symmetric monoidal categories.

2.1. Facultative properties of tensor categories.

DEFINITION 2.1.1. A tensor category C has $\underline{\mathrm{Hom}}$-s (inner homomorphisms) and called *inner* if for any two objects $U, V \in Ob(C)$ there exists an object $\underline{\mathrm{Hom}}(U,V)$ and a functorial in $W \in Ob(C)$ isomorphism

$$\mathrm{Hom}(W, \underline{\mathrm{Hom}}(U,V)) \simeq \mathrm{Hom}(W \otimes U, V))$$

If C has $\underline{\mathrm{Hom}}$-s then the construction $(U,V) \mapsto \underline{\mathrm{Hom}}(U,V)$ can be canonically amde into a bilinear functor $C^{op} \times C \longrightarrow C$. Applying the universal property to the case $W := \underline{\mathrm{Hom}}(U,V)$ mapped identically to itself, we get canonical morphism

$$U \otimes \underline{\mathrm{Hom}}(U,V) \longrightarrow V$$

EXAMPLE 2.1.2. In the category $Rep_{k,G}$, for two representations U and V vector space $\mathrm{Hom}(U,V)$ is the space of interwinning operators. Object $\underline{\mathrm{Hom}}(U,V)$ is the representation of G in the space $\mathrm{Hom}(forget(U), forget(V))$ where

$$forget : Rep_{k,G} \longrightarrow Vect_k$$

is the functor associating with a representation the underlying vector space. Notice that we have a canonical identification

$$\mathrm{Hom}(U,V) \simeq \mathrm{Hom}(\mathbf{1}, \underline{\mathrm{Hom}}(U,V))$$

which holds in fact in general tensor category.

Tensor categories with $\underline{\mathrm{Hom}}$-s and infinite sums and products behave in almost all respects exactly as the category $Vect_k$.

DEFINITION 2.1.3. Let $\mathcal{C}$ be an inner tensor category. An object U of $\mathcal{C}$ is called *finite* iff there exists an object U^* and a homorphism $U \otimes U^* \longrightarrow \mathbf{1}$ such that for any $V \in ObC$ induced maps

$$U^* \otimes V \longrightarrow \underline{\mathrm{Hom}}(U, V), U \otimes V \longrightarrow \underline{\mathrm{Hom}}(U^*, V)$$

are isomorphisms.

It follows immediately that U^* is functorially isomorphic to $\underline{\mathrm{Hom}}(U, \mathbf{1})$. For example, in tensor category $Rep_{k,G}$ finite objects are exactly finite-dimensional representations. We adopt notation $U^* := \underline{\mathrm{Hom}}(U, \mathbf{1})$ also for non-finite objects in tensor categories with $\underline{\mathrm{Hom}}$-s.

For a finite U we have canonical maps

$$U \otimes U^* \longrightarrow \mathbf{1} \text{ and } \mathbf{1} \longrightarrow U^* \otimes U$$

The first morphism comes from the evaluation morphism $U \otimes \underline{\mathrm{Hom}}(U, \mathbf{1}) \longrightarrow \mathbf{1}$, and the second morphism comes from the identity morphism

$$id_U \in \mathrm{Hom}(U, U) = \mathrm{Hom}(\mathbf{1}, \underline{\mathrm{Hom}}(U, U)) = \mathrm{Hom}(\mathbf{1}, U^* \otimes U)$$

Thus, we can form the composition

$$\mathbf{1} \longrightarrow U^* \otimes U \simeq U \otimes U^* \longrightarrow \mathbf{1}$$

which is an element of $\mathrm{Hom}(\mathbf{1}, \mathbf{1}) = k$ called the *rank* of V and is denoted by $rank\,(V)$.

EXAMPLE 2.1.4. In the category of vector spaces the rank $rank\,(V)$ coincides with the usual dimension $dim\,(V) \in \mathbf{N}$ considered as an element of k.

DEFINITION 2.1.5. An inner tensor category is called rigid if all its objects are finite.

An example of a rigid tensor category is the category of finite-dimensional vector spaces, or of finite-dimensional representations of a group.

3. Examples of tensor categories

3.1. Classical examples. In classical examples objects are vector spaces endowed with an additional structure, and the ternsor product, commutativity and associativity morphisms are the same as in the the category of vector spaces. In other words, tensor category $\mathcal{C}$ is considered together with a faithful symmetric monoidal functor $F : C \longrightarrow Vect_k$. Such a functor is called *fiber functor*.

- the basic example: tensor category $Vect_k$ of vector spaces,
- the category $Rep_{k,G}$ of k-linear representations of an abstract group G,
- category $A - mod$ of modules over a cocommutative bialgebra A over k,
- category $A - comod$ of comodules over a commutative bialgebra A.

The second example contains in the third: one takes A be equal to the group algebra $k[G]$ of an abstract group G.

Notice that if k is not algebarically close, there could be several non-isomorphic fiber functors for the same tensor category $\mathcal{C}$. For example, if G_1, G_2 are two affine algebraic groups over k which are inner twisted forms of each other, then Rep_{k,G_1} is equivalent to Rep_{k,G_2}.

3.2. Supervector spaces. Tensor category $Super_k$ of supervector spaces over k is defined as follows: as monoidal category it is identified with the category $Rep_{k,\mathbf{Z}/2\mathbf{Z}}$ of representations of $\mathbf{Z}/2\mathbf{Z}$, i.e. $\mathbf{Z}/2\mathbf{Z}$-graded vector spaces. The commutativity morphism

$$comm : V \otimes U \longrightarrow U \otimes V$$

on homogeneous elements is

$$comm(v \otimes u) = \begin{cases} -u \otimes v & \text{if both } u \text{ and } v \text{ are odd,} \\ u \otimes v & \text{otherwise} \end{cases}$$

The check of axioms of a tensor category is straightforward. If a choice of the ground field k is clear, we will skip it from the notation.

Another way is to define $Super$ using the axiomatics of acylic tensor calculus. Namely, the space $\mathrm{Hom}((V_i)_{i\in I}, (U_j)_{j\in J})$ for two finite collections of $\mathbf{Z}/2\mathbf{Z}$-graded vector spaces can be defined as

$$\bigoplus_{\substack{\epsilon: I \longrightarrow \{0,1\} \\ \epsilon': J \longrightarrow \{0,1\} \\ \sum \epsilon(i) = \sum \epsilon'(j) (mod 2)}} \mathrm{Hom}_{Vect}(\otimes_{i\in I} V_i^{\epsilon(i)}, \otimes_{j\in J} U_j^{\epsilon'(j)}) \otimes D(\epsilon, \epsilon')$$

where $D(\epsilon, \epsilon')$ is one dimensional vector space equal to the top exterior power of

$$k^{\{i\in I | \epsilon(i)=1\} \sqcup \{j \in J | \epsilon'(j) = 1\}}$$

There are no artificial signs in the definition of tensor products and compositions of polymorphisms.

The tensor category $Super_k$ is "almost" the representations of $\mathbf{Z}_2$:

EXERCISE 3.2.1. Constructions of categories $Rep_{k,G}$ where G is a group, or $Super_k$ can be performed in arbitrary tensor category. Check that the category of supervector spaces in $Super_k$ is equivalent to the category of representations of $\mathbf{Z}/2\mathbf{Z}$ in $Super_k$.

3.3. Z-graded spaces, complexes. (4) The category $Vect_k^{\mathbf{Z}}$. This is a generalization of the previous example. Objects of the category $Vect_{\mathbf{Z}}$ are infinite sums $V = \oplus_{n\in\mathbf{Z}} V^n$ of k-vector spaces. We assign the grading n to all elements of V^n. The tensor product is the natural one: for two graded spaces V, W we define $V \otimes W = \oplus_n U^n$, where $U^n = \oplus_{i+j=n} V^i \otimes W^j$. Then the associativity constraint is given by the identity map, and the commutativity morphism is completely determined on the graded components: $V^n \otimes W^m \to W^m \otimes V^n$ is given by the flip map multiplied by $(-1)^{nm}$. One can check that in this way we obtain a tensor category with the unit object $\mathbf{1}$ given by the graded space which has all zero components, except $\mathbf{1}^0 = k$. Sometimes we will denote it simply by k.

For a graded vector space V and an integer i we denote by $V[i]$ the new graded vector space, such that $(V[i])^n = V^{n+i}$.

Let $k[-1]$ denotes the graded vector space $\mathbf{1}[-1]$. Then $V[-i] = V \otimes k[-1]^{\otimes i}$. To be pedantic, we will sometimes write $V = \oplus_{n\in\mathbf{Z}} V^n[-n]$. This means that we consider the graded components V^n as vector spaces having degrees zero. We hope such a notation will not lead to a confusion.

(5) The category of complexes $\mathbf{K}$ of vector spaces is a tensor category. We leave to the reader to work out the details similarly to the previous example. There are natural tensor functors $\mathbf{K} \to Vect_k^{\mathbf{Z}}$ (forgetful functor), $Vect_k^{\mathbf{Z}} \to Super_k$ (all V^{2n} receive degree zero, all V^{2n+1} receive degree one).

There are plenty of geometric examples of symmetric monoidal and tensor categories.

(6) The category of topological spaces with the operation of disjoint union as a tensor product, and the empty space as the unit object. One can make similar categories of smooth manifolds, algebraic varieties, smooth projective varieties, etc. All those are symmetric monoidal categories.

(7) The category of topological spaces with the Cartesian product as a tensor product is a symmetric monoidal category. There is no canonically defined unit object. But all unit objects (points) are naturally isomorphic.

An important class of tensor categories consists of semi-simple ones.

Each object of a semi-simple category is, by definition, a finite sum of simple objects. A tensor category is called semi-simple if it is semi-simple as a category.

If $\mathcal{C}$ is a semi-simple tensor category, then the tensor product of two objects $X \otimes Y$ is a finite sum of some other objects.

The category of vector spaces $Vect_k$ and all its cousins (like $Super_k$, $Vect_k^{\mathbf{Z}}$, $\mathbf{K}$ are semi-simple. Hovewer, there are many tensor categories which are not semi-simple. For example $Rep_{k,G}$ is not always semisimple (it is such, if G is a reductive group).

Exercise 3.3.1. Define a tensor product of two semi-simple tensor categories over a field of characteristic zero in such a way that the tensor product of the representation categories of two finite groups becomes the representation category of their product.

Then show that

$\mathrm{Super}_k \otimes \mathrm{Rep}_{k,\mathbf{Z}_2} = \mathrm{Super}_k \otimes \mathrm{Super}_k$.

This result means, that in some sense, $Super_k$ is the representations of a "twisted form of $\mathbf{Z}_2$."

An analog of the category of finite-dimensional spaces is given by *rigid* tensor categories.

A rigid tensor category is a tensor category $\mathcal{C}$ together with a duality functor $* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ together with functorial morphisms $\mathbf{1} \to V \otimes V^*, V^* \otimes V \to \mathbf{1}$. There are natural axioms for these morphisms. It is also required that any object is a dual to some. It implies that there is a functorial isomorphism $V \to V^{**}$.

The rigidity of $\mathcal{C}$ gives rise to a map *rank*: $Ob\,C \to k = \mathrm{Hom}(\mathbf{1},\mathbf{1})$. First we define the trace map Tr_V of as a composition $Hom(V,V) \to Hom(\mathbf{1}, V^* \otimes V) \to Hom(\mathbf{1},\mathbf{1}) = k$. Then we define $rank(V)$ as $Tr_V(id_V)$. It is easy to see that $rank(V \otimes W) = rank(V)rank(W)$ and $rank(\mathbf{1}) = 1$. In $Vect_k$ we have: $rank(V) = dim(V)$.

In the rigid tensor category of supervector spaces, the rank of (V_0, V_1) is $\mathrm{dim}V_0 - \mathrm{dim}V_1$. We will call the pair $(dimV_0|dimV_1)$ the *superdimension* of V.

Since the rank can be negative, the category $Super_k$ is not equivalent to the category $Rep_{k,G}$ for any G.

The following theorem was proved by Deligne (see his paper in Grothendieck Festschrift, vol.2).

THEOREM 3.3.2. *Let k be an algebraically closed field of characteristic zero, $\mathcal{C}$ a rigid tensor category. If ranks of all objects are non-negative integers then there is a fiber functor*

$F : \mathcal{C} \to \mathrm{Vect}_k$ *(this means that F is faithful and commuting with the tensor structures).*

COROLLARY 3.3.3. *There is a commutative Hopf algebra A over k such that $\mathcal{C}$ is the category of comodules over A.*

Having this result we can in fact reconstruct an affine pro-algebraic group G, such that A is the algebra of functions on it. Roughly speaking, G is the group $Aut(F)$ of automorphisms of F as a *tensor* functor.

Let us say few more words about the proof and the reconstruction of G. Since the latter can be infinite-dimensional, one has to be careful. First one takes $End(F)$, which is the Hopf algebra of endomorphisms of the tensor functor F. It is easy to see that it is a cocommutative Hopf algebra. Thus the dual to it is the algebra we need. The rest of the proof follows from the general fact about commutative Hopf algberas.

PROPOSITION 3.3.4. *Then A is an inductive limit of A_α, where A_α is finitely generated, i.e. functions on an affine scheme of finite type which is in fact an algebraic group.*

Thus $\mathcal{C}$ is the category of representations of an affine pro-algebraic group.

Deligne and Milne gave an example of a rigid tensor category in which the rank takes noninteger values. This semisimple rigid abelian category is denoted by GL_t. It can be considered as a "continuation" of the category $Rep_{\mathbf{Q},GL_n}$ of the finite-dimensional representations of the linear group GL_n to non-integer n. Base field for GL_t is the field of rational functions $\mathbf{Q}(t)$. There is an object T in the category with the rank equal to t. Therefore GL_t is not a category of the type $Rep_{k,G}$. More details will be given in the next subsection.

We would like to finish this subsection with the following

CONJECTURE 3.3.5. Rigid tensor categories with ranks in $\mathbf{Z}$ can be of two types: comodules over commutative Hopf algebras or comodules over supercommutative Hopf algebras.

4. Tensor category GL_t

Let us consider the category $\mathcal{A}$ such that its objects are : empty set, or finite collections S of 0-dimensional oriented manifolds together with a partition $S = S_+ \cup S_-$ into two subsets. In other words, objects are pairs $(m,n) \in \mathbf{Z}_+^2$. We define $Hom(S_1, S_2)$ as the set of classes of diffeomorphisms of oriented 1-dimensional manifolds L such that $\partial L = S_1 \cup S_2$. We equip the category with a symmetric monoidal structure, which is induced by the operation of disjoint union of objects. The unit object $\mathbf{1}$ corresponds to the empty set with the identity morphism. We can make $\mathcal{A}$ into a $\mathbf{Q}$-linear category, taking formally $\mathbf{Q}$-linear combinations of morphisms. Then $End(\mathbf{1})$ is isomorphic to $\mathbf{Q}[t]$ where t is the diffeomorphism class of the unit circle S^1. Moreover, $\mathcal{A}$ is a rigid tensor category with $(m,n)^* = (n,m)$ (taking the opposite orientation).

Suppose that $\mathcal{C}$ is a rigid tensor category, $V \in Ob(\mathcal{C})$. Then there is a tensor functor $F : \mathcal{A} \to \mathcal{C}$ such that $F(m,n) = V^{\otimes m} \otimes V^{*\otimes n}$. We will denote the RHS of this formula by $T^{(m,n)}(V)$. Let us take $\mathcal{C}$ to be the category $Vect^f$ of finite-dimensional $\mathbf{C}$-vector spaces. Then for fixed m,n, and $rk(V) = dim_{\mathbf{C}}(V) \geq n+m$ we have: $End(T^{(m,n)}(V)) \simeq \mathbf{C}[S_{m+n}]$ (group algebra of the symmetric group). In particular , it does not depend on V. It implies that the algebra $A_{m,n} = End(T^{(m,n)}(V))$ is semisimple (finite sum of matrix algebras). We can extend $\mathcal{A}$ in two steps: a) extending scalars to rational functions $\mathbf{Q}(t)$, so that $End(m,n)$ is replaced by $End(m,n) \otimes \mathbf{Q}(t)$; b) adding idempotents corresponding to the projectors to the irreducible components of $T^{(m,n)}(V)$ (the latter extension is called Karoubian envelope). The resulting tensor category is called GL_t. It contains objects with the rank which is not an integer, but a rational function in t.

5. Signs and orientations

6. Applications of supermathematics

6.1. Identification of symplectic and orthogonal geometry. Let V be a supervector space, B a bilinear form on V with values in $\mathbf{1}$=$\mathbf{k}$ (the unit object in the tensor category $Super_k$). Then we can apply the functor of changing the parity $\Pi V = V \otimes k^{0|1}$. In this way we get a new bilinear form $\tilde{B}$ on ΠV. It is given by $\tilde{B} = B \otimes \nu$, where $\nu : k^{0|1} \otimes k^{0|1} \to \mathbf{1}$ is the bilinear form such that $\nu(a\varepsilon, b\varepsilon) = ab, a, b \in k$, ε is the fixed base element of $k^{0|1}$. If B is a skew-symmetric form than $\tilde{B}$ is symmetric (in graded sense).

Then we have the following informal observation: $Sp(2n) \simeq O(-2n)$.

We are going to interpret this isomorphism in purely classical terms (i.e. without supermathemtics).

Let g be a Lie subalgebra of $gl(V)$, where $V$is a finite-dimensional vector space. Suppose that the bilinear form $\mathrm{tr}(XY)$ is nondegenerate on g. This leads to many numerical invariants of g as follows. Let us choose an orthonormal base $\{X_i\}$ of g. Then the structure constants c_{ijk} of g in this base are totally skew symmetric.

Now let us fix a word in some alphabet, and divide it into three letter subwords. Suppose that each letter appears twice in the word. For instance: *ijk jik*. Then we can construct the sum

$$I = \sum_{i,j,k} c_{ijk} c_{jik}.$$

This number is independent of the choice of orthonormal basis.

For example if g is semisimple Lie algebra then it carries the Killing form $\langle x,y \rangle = tr(adx \cdot ady)$. Then in the orthonormal base as above we can write for the trace of the identity operator acting in g:

$$dimg = \sum_i \langle X_i, X_i \rangle = \sum_i tr(adX_i)^2 = \sum_{i,j,k} c_{ijk} c_{ikj}$$

Hence $I = dimg$ depends on g only.

All such words are labeled by trivalent graphs (vertex = subword, edge = letter).

Now look at the algebras

$$\begin{array}{cccccccc} -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ Sp(4) & 0 & Sp(2) & 0 & 0 & 0 & o(2) & o(3) \end{array}$$

EXERCISE 6.1.1. Any of the invariants above is given by the values of a polynomial in n.

EXAMPLE 6.1.2. Dimension of $o(n) = n(n-1)/2$, of $sp(m)$ is $m(m+1)/2$.

The best solution of this problem uses the functor Π in supermathematics, if one observes that $o(1-2n) \simeq osp(1, 2n)$. More generally, one could also look at $osp(n|2m)$. This is a Lie superalgebra defined by a nondegenerate even bilinear form.

6.2. Where does the De Rham complex come from? The following idea has been already discussed.

Let $A^{0|1}$ be the superscheme whose function ring is the symmetric algebra $S((k^{0|1})^*) = k^{1|1} = k[\epsilon]$ where ϵ is an odd variable ($\epsilon^2 = 0$).

$Aut(A^{0|1})$ is the function algebra of a supergroup scheme of automorphisms of $A^{0|1}$. Its comodules are $\mathbf{Z}$-graded complexes.

On a manifold X, we have the scheme of maps from $A^{0|1}$ to X. The automorphism group of $A^{0|1}$ acts on it. We have shown that it leads to the De Rham complex of X.

7. Pseudo-tensor categories, operads, PROPs

In [BD] the notion of *pseudo-tensor category* was introduced as a generalization of the notion of symmetric monoidal (=tensor) category. Similar notion was introduced by Borcherds under the name *multi-linear category.*

This notion is essentially equivalent to the notion of colored operad (see Chapter 5).

DEFINITION 7.0.1. A pseudo-tensor category is given by the following data:

1. A class $\mathcal{A}$ called the class of objects, and a symmetric monoidal category $\mathcal{V}$ called the category of operations.

2. For every finite set I, a family $(X_i)_{i\in I}$ of objects, and an object Y, an object $P_I((X_i), Y) \in \mathcal{V}$ called the space of operations from $(X_i)_{i\in I}$ to Y.

3. For any map of finite sets $\pi : J \to I$, two families of objects $(Y_i)_{i\in I}$, $(X_j)_{j\in J}$ and an object Z, a morphism in $\mathcal{V}$

$$P_I((Y_i), Z) \otimes (\otimes_i P_{\pi^{-1}(i)}((X_{j_i}), Y_i)) \to P_J((X_j), Z)$$

called composition of operations. Here we denote by $\otimes$ the tensor product in $\mathcal{M}$.

4. For an 1-element set $\bullet$ and an object X, a unit morphism $\mathbf{1}_{\mathcal{V}} \to P_\bullet((X), X)$.

These data are required to satisfy natural conditions. In particular, compositions of operations are associative with respect to morphisms of finite sets, and the unit morphisms satisfy the properties analogous to those of the identity morphisms (see [BD96] for details).

If $\mathcal{A}$ is a set, then a pseudo-tensor category is exactly the same as an $\mathcal{A}$-colored operad in the tensor category $\mathcal{V}$.

If we take $\mathcal{V}$ to be the category of sets, and take I above to be 1-element sets only, we obtain a category with the class of objects equal to $\mathcal{A}$.

Pseudo-tensor category with one object is called an *operad.* We will study operads in Chapter 5.

A symmetric monoidal category $\mathcal{A}$ (see Appendix) produces the symmetric monoidal category with $P_I((X_i), Y) = Hom_{\mathcal{A}}(\otimes_i X_i, Y)$.

The notion of pseudo-tensor category admits a generalization to the case when no action of symmetric group is assumed. This means that we consider *sequences* of objects instead of *families* (see [So99]). The new notion generalizes monoidal categories.

Finally, we mention that small symmetric monoidal categories are closely related to PROPs. One can describe PROPs similarly to pseudo-tensor categories. Namely a k-linear PROP is given by a class of objects $\mathcal{A}$, and for any finite sets I, J, families of objects $(X_i)_{i\in I}, (Y_j)_{j\in J}$ a vector space $P_{I,J}((X_i)_{i\in I}, (Y_j)_{j\in J})$. Collections $P_{I,J}$ satisfy natural properties which generalize those for k-linear pseudo-tensor categories. For example, a PROP with one object X is determined by the collection of sets $P_{n,m} = Hom(X^{\otimes n}, X^{\otimes m})$, as well we natural compositions between them.

PROPs form a category with the naturally defined morphisms.

The following result describes the relationship of PROPs with tensor categories.

PROPOSITION 7.0.2. *Category of PROPs is equivalent to the category formed by objects given by the data a) and b) below:*

a) it is a tensor category $\mathcal{C}$ such that $Ob(\mathcal{C}) = (E_0, ..., E_n, ...)$ are in one-to-one correspondence with non-negative integers.

b) isomorphisms $E_n \simeq E_1^{\otimes n}$ (in particular, $E_0 \simeq \mathbf{1}$).

Proof. We define $P_{\{1,...,n\},\{1,...,m\}} = Hom_{\mathcal{C}}(E_n, E_m)$. It is easy to check that this is the desired equivalence. ■

Let I be a set. One defines I-colored PROPs similarly to the ordinary PROPs replacing finite sets by finite I-sets. Recall that a finite I-set is a finite set J plus a map $J \to I$. We leave to the reader to work out the definition of the I-colored PROP and prove, that to have an I-colored PROP is the same as to have a tensor category $\mathcal{C}$ with objects $X_{n_1,...,n_k}, n_i \geq 0$ such that introducing objects $X_{e_i}, i \geq 0, e_i = (0, ..., 1, ..., 0)$ (1 on the ith place) one has isomorphisms $X_{n_1,...,n_k} \simeq X_{e_1}^{\otimes n_1} \otimes ... \otimes X_{e_k}^{\otimes n_k}$.

PROPs can be used to encode data of linear algebra.

EXAMPLE 7.0.3. Let G be an affine group scheme over a field k, $V = \mathcal{O}(G)$ algebra of regular functions on G. Then we have a PROP generated by the following data:

a) $k \to V$ (unit);

b) $V \otimes V \to V$ (product);

c) $V \to V \otimes V$ (coproduct);

d) $V \to V$ (antipode).

These data are subject to the well-known Hopf algebra axioms (associtivity of the product, coassociativity of the coproduct, etc.).

In this way we obtain a PROP, which encodes in terms of linear algebra the affine group structure.

EXAMPLE 7.0.4. Suppose that an affine group G acts on an affine scheme X. Then, in addition to a)-d) from the previous example, we have a)-b) for $W = \mathcal{O}(X)$ as well as the homomorphism of algebras $W \to V \otimes W$ which encodes the group action. These data satisfy well-known axioms, which gives rise to a PROP, which encodes in terms of linear algebra the group action.

EXAMPLE 7.0.5. Let V be a vector space over k. Then we have the endomorphism PROP $End(V)$ such that $End(V)(n,m) = Hom(V^{\otimes n}, V^{\otimes m})$. Using this example one can define a representation of an arbitrary PROP H as a morphism of PROPs $H \to End(V)$. Sometimes, we will say that V is an algebra over the PROP H.

Concerning the last example, we remark that for an I-colored PROP H one can say what is the representation of H in $Vect_k$. In this case one assigns an object $V \in Vect_k$ to each color (i.e. element of I). It is easu to see that to have a representation of H in $Vect_k$ is teh same as to have a tensor functor $\mathcal{C} \to Vect_k$, where $\mathcal{C}$ is the tensor category associated to the PROP H (see above).

8. Supermanifolds

8.1. Definitions. So far we have been doing linear algebra in tensor categories. Our main example was the category of supervector spaces and its immediate generalizations. We would like to do some differential and algebraic geometry within the same framework. For example we want to have "manifolds" which locally look as commutative superalgebras. We do not want to go into detailed discussion of the topic. There are several books devoted to what is called "supergeometry". We briefly recall main ideas.

We use with the standard convention about signs which was suggested by Quillen. Namely, we write $\pm$ for

$$(-1)^{\text{sign of permutation of odd symbols}},$$

and $\mp$ for $-\pm$.

For example, in a super Lie algebra case we write $[x,y] = \mp[y,x]$, $[x,[y,z]] = [[x,y],z] \pm [y,[x,z]]$.

DEFINITION 8.1.1. A supermanifold of superdimension $n|m$ is a topological space X equipped with a sheaf of topological supercommutative associative algebras with unit, which is locally isomorphic to the standard model $\mathbf{R}^{n|m}$. This means that its underlying space is $\mathbf{R}^n$, and the functions on an open subset are elements of $C^\infty(U) \otimes S((\mathbf{R}^{0|m})^*)$ (we write it this way rather than as a wedge product).

Then locally X has n "even" and m "odd" coordinates.

THEOREM 8.1.2. *Every $n|m$-dimensional supermanifold Y is isomorphic to the one coming from a vector bundle V of rank m on an ordinary n-dimensional manifold X. The latter defines a supermanifold for which the functions are sections of the wedge powers of V^*.*

We denote the algebra of functions on a supermanifold X by $O(X)$. It is clear how to make supermanifolds into a symmetric monoidal category. For this we define a morphism of a category of supermanifolds. A morphism $X \to Y$ in this category is a morphism of ringed spaces. Then (locally) it is a pair (ϕ, ψ) such that ϕ is a morphism of underlying usual (even) smooth manifolds and $\psi : O(Y) \to O(X)$ is a homomorphism of supercommutative algebras with unit such that $\psi(f)(x) = f(\phi(x))$ for any $f \in O(Y), x \in X$.

This definition works for superschemes as well.

EXERCISE 8.1.3. (on composition of maps). Consider $\mathbf{R}^{1|2k}$, mapped to $\mathbf{R}$ by the formula

$$y = x + \xi_1\eta_1 + ... + \xi_k\eta_k.$$

Here x is the even coordinate and ξ_i, η_i are odd coordinates.

Now let $z = \sin(y)$. What is $z(x, \xi, \eta)$?

DEFINITION 8.1.4. Supervector bundle over supermanifold Y is a sheaf of O_Y-modules which is locally free and finitely generated (i.e.locally is $O_Y \otimes \mathbf{R}^{k|l}$).

If V is a super vector bundle, we denote by tot$V$ is its total space considered as a supermanifold.

We have all standard operations on vectors bundles over supermanifolds:

direct sum, tensor product, dual, *change of parity* operator Π (defined as the tensor product with $\mathbf{R}^{0|1}$).

There are four bundles naturally associated with a supermanifold Y:

$$T_Y Y, \Pi T_Y, T_Y^*, \Pi T_Y^*.$$

EXERCISE 8.1.5. 1. Define a structure of Lie superalgebra on the sections of T_Y.

2. Define an odd vector field D on the total space ΠTY of ΠT_Y such that $[D, D] = 0$. Note that the functions on ΠTY are called differential forms on Y.

There are 3 versions of differential forms. Let x_i, ξ_j be coordinates on Y.

(a) all C^∞ functions in $D\xi_j$;

(b) all polynomials in $D\xi_j$;

(c) all distributions in $D\xi_j$.

We will use only the choice (b) (If Y is an ordinary manifold, this problem does not arise.)

3. Define a closed (even) non-degenerate 2-form ω on T^*Y. Its inverse is a bivector field on T^*Y, which gives a Poisson bracket on functions on T^*Y, making them a Lie superalgebra.

If Y is even then it is the standard symplectic structure on T^*Y.

4. Define an *odd* closed 2 form on ΠT^*Y to get an *odd* Poisson structure, and get again a Lie superalgebra structure which in the case where Y is even is the Schouten bracket on the polyvector fields.

REMARK 8.1.6. In the presence of odd coordinates, one cannot integrate differential forms. One can see this by looking at changes of coordinates. To solve this problem one introduces a new concept of integral. It is called Berezin integral. The new theory ofintegration requires "integral forms" which can be integrated, but not multiplied.

8.2. Superschemes. Here we briefly recall how to define the notion of a scheme in the framework of supermathematics. Most of what we say should work in arbitrary tensor category.

Let us recall the standard case. Affine schemes over k are commutative associative algebras with unit, but with arrows reversed. We denote by $O(S)$ the algebra of functions on an affine scheme S. We denote by $Spec(A)$ (or $SpecA$) the affine scheme corresponding to a commutative k-algebra A.

There is a notion of L-points of an affine scheme $Spec(A)$ where L is another commutative unital k-algebra. By definition points are homomorphisms of unital k-algebras $A \to L$. In particular k-points of $Spec(A)$ are algebra homomorphisms $A \to k$,

One can "superize" the above definitions in the obvious way using supercommutative algebras instead of commutative ones. What we get is the notion of affine superscheme.

EXAMPLE 8.2.1. Let V be a supervector space. Consider $S(V)$, which is the direct sum of symmetric powers of V. The latter is defined as the coinvariants of the (super) action of the symmetric groups on the tensor powers of V.

Notation: when $dimV = n|m$, we write $Spec(S(V^*)) = A^{n|m}$.

A general finitely generated affine superscheme corresponds to the quotient of such an algebra by a $\mathbf{Z}_2$-graded ideal.

8.3. Diffeomorphisms of $0|1$-dimensional space. Let $S = \mathrm{Maps}(A^{0|1}, A^{0|1})$. We put $A = B = O(A^{0|1}) = k^{1|1}$. Let ξ be the odd coordinate on $A^{0|1}$. For $f \in S$ we have: $f(\xi) = a + b\xi$. The generators are $a = a(f)$ (odd) and $b = b(f)$ (even).

The function ring $O(S)$ is isomorphic to $k[b] \otimes k[a]/(a^2)$.

Composition of functions gives us the coproduct on this algebra. If $f_1(\xi) = a_1 + b_1\xi$ and $f_2(\xi) = a_2 + b_2\xi$ then $f_1(f_2(\xi)) = a_2 + b_2a_1 + b_1b_2\xi$. Therefore:

$\Delta(b) = b \otimes b$,

$\Delta(a) = a \otimes 1 + b \otimes a$.

Let us denote by S^* the set of automorphisms of $A^{0|1}$, i.e. invertible elements of S.

This is a closed supersubscheme of $S \times S$ (pairs of automorphisms with their inverses). Clearly S^* is a group object in the category of superschemes. Then $O(S^*)$ is a Hopf algebra.

We write $S^* = G_m \times G_a$, where G_m is $\mathrm{Spec}[b, b^{-1}]$ and G_a is $A^{0|1}$.

8.4. Representations of the group scheme S^*. A representation of S^* is a supervector space V with a right comodule structure $\rho : V \to V \otimes O(S^*) = V \otimes k[b, b^{-1}, a]/(a^2)$.

We can write it as $v \mapsto \sum_n (P_n(v) \otimes b^n \pm Q_n(v) \otimes ab^n)$, where almost all $P_n(v)$ and $Q_n(v)$ are equal to zero for any given v. Here P_n, Q_n are linear morphisms of V.

Now we need commutativity of some diagrams to specify that we have a coalgebra action (compatibility with coproduct and counit). These give rise to identities for P_n and Q_n. Namely, let Δ be the coproduct of the Hopf superalgebra $O(S^*)$. Then $\Delta(b) = b \otimes b, \Delta(a) = a \otimes 1 + b \otimes a$. Compatibility of ρ with Δ means that $(id \otimes \Delta)\rho = (\rho \otimes id)\rho$. Then we have

$$(id \otimes \Delta)\rho(v) = \sum_n P_n(v) \otimes b^n \otimes b^n \pm \sum_n Q_n(v) \otimes (ab^n \otimes b^n + b^{n+1} \otimes ab^n)$$

and

$(\rho \otimes id)\rho(v) = \sum_{m,n}(P_m P_n(v) \otimes b^m \otimes b^n \pm P_m Q_n(v) \otimes ab^m \otimes b^n \pm Q_m P_n(v) \otimes b^m \otimes ab^n + Q_m Q_n(v) \otimes ab^m \otimes ab^n)$.

Comparing these equations we obtain:

$P_n P_m = \delta_{nm} P_n$, $P_m Q_n = \delta_{m,n+1} Q_n$, $Q_m P_n = \delta_{mn} Q_n$, $Q_m Q_n = 0$.

We also remark that the coaction ρ is given by the formula $f(v) \mapsto f(gv)$ where g is an element of the supergroup. In particular if g is the unit element then $f(gv) = f(v)$. It corresponds to $a = 0, b = 1$ in the formulas for ρ, and hence $\sum_n P_n(v) = v$ for any vector v (equivalently it follows from the diagram for the compatibility of ρ and the counit).

From these considerations we obtain the following equations for P_n:

$P_k \circ P_l = 0,\ k \neq l$,

$P_n \circ P_n = P_n$,

$\sum_n P_n = \mathrm{Id}_V$.

In other words, we have commuting projections which give a direct sum decomposition of V making it into a $\mathbf{Z}$-graded vector space $V = \oplus_k V^k$.

We also conclude that Q_k maps V^k to V^{k+1} and $Q_k^2 = 0$.

So we get exactly complexes!

Remark 8.4.1. *"Correct category"* which arises in practice is not the full tensor category of complexes of supervector spaces, but its tensor subcategory, consisting of those complexes, for which V^{even} is even and V^{odd} is odd. If we forget the differential of a complex, then we obtain an object of the tensor category $Vect_k^{\mathbf{Z}}$.

8.5. Another remark on the origin of De Rham complex. Let X be an affine superscheme. It is easy to see that $\Pi TX = \mathrm{Maps}(A^{0|1}, X)$. Then $O(\Pi TX)$ is the algebra generated by a and da, for $a \in O(X)$, with relations given by those in the ordinary algebra of functions, together with $d(ab) = (da)b \pm a(db)$.

By general nonsense, the scheme $S^* = \mathrm{Aut}(A^{0|1})$ acts on $Maps(A^{0|1}, X)$ making it into a differential graded algebra. In this way we obtain the algebra of differential forms on X.

9. Appendix: Symmetric monoidal categories

9.0.1. *A glimpse of the classical definition.* Tensor category is a "linear" version of a more general notion of symmetric monoidal category. A symmetric monoidal category consists of a category $\mathcal{C}$, a functor $\otimes : C \times C \longrightarrow C$, the unit object $\mathbf{1} \in Ob(C)$, and associativity, commutativity and unit morphisms as in the definition 2.0.1 satysfying certain coherence axioms (see [MacLane]???). These axioms (pentagon axiom, hexagon axiom etc.) are quite complicated.

Here we will use a different approach which seems to be more transparent. The idea is to change slightly the data putting inside all possible universal constructions.

Later we will see an analogous of algebraic structures in terms of operads (see Chapter 5).

Definition 9.0.1. A symmetric monoidal category $\mathcal{C}$ is the following data:

(1) a class $Ob(\mathcal{C})$ called the class of objects
(2) a set $\mathrm{Hom}((X_1, \ldots, X_n), (Y_1, \ldots, Y_m))$ (of polymorphisms) for any non-negative integers $n, m \geq 0$, objects $X_1, \ldots, X_n \in Ob(C)$ and $Y_1, \ldots, Y_m \in Ob(C)$

(3) a bijection

$$: i_{\sigma,\sigma'} : \mathrm{Hom}((X_1,\ldots,X_n),(Y_1,\ldots,Y_m)) \simeq \mathrm{Hom}(X_{\sigma(1)},\ldots,X_{\sigma(n)};(Y_{sigma'(1)},\ldots,Y_{\sigma'(m)})$$

for any permutations $\sigma \in \Sigma_n, \sigma' \in \Sigma_m$ and any $(X_i),(Y_j) \in Ob(C)$

(4) an element $id_X \in \mathrm{Hom}(X;X)$ for any $X \in Ob(C)$

(5) a map (called composition)

$$\mathrm{Hom}((X_1,\ldots,X_n),(Y_1,\ldots,Y_m)) \times \mathrm{Hom}((X'_1,\ldots,X'_{n'}),(Y'_1,\ldots,Y'_{m'}))$$

$$\longrightarrow \mathrm{Hom}((X_1,\ldots,X_n,X'_2,\ldots,X'_{n'}),(Y-2,\ldots,Y_n,Y'_1,\ldots,Y'_{m'})$$

for any $n, m' \geq 0, n', m \geq 1$ and any $(X_i),(X'_i),(Y_i),(Y'_i)$ such that $X'_1 = Y_1$

(6) a map (called tensor product of polymorphisms)

$$\mathrm{Hom}((X_1,\ldots,X_n),(Y_1,\ldots,Y_m)) \times \mathrm{Hom}((X'_1,\ldots,X'_{n'}),(Y'_1,\ldots,Y'_{m'}))$$

$$\longrightarrow \mathrm{Hom}((X_1,\ldots,X_n,X'_1,\ldots,X'_{n'}),(Y_1,\ldots,Y_m,Y'_1,\ldots,Y'_{m'}))$$

satisfying axioms

(1) $\imath_{\sigma_2,\sigma'_2} \circ \imath_{\sigma_1,\sigma'_1} = \imath_{\sigma_2\circ\sigma_1,\sigma'_2\circ\sigma'_1}$, which means that we can associate canonically the set $\mathrm{Hom}((X_i)_{i\in I},(Y_j)_{j\in J})$ with any finite collections $(X_i)_{i\in I},(Y_j)_{j\in J}$ of objects,

(2) composition of any polymorphism f with the identity morphism on the left and on the right coincides with f,

(3) (associativity of compositions) Let Γ be an oriented acyclic graph and for each edge $e \in E(\Gamma)$ we choose an object $X_e \in Ob(C)$. Using numerations of some sets of edges, and compositions and tensor prodcuts in some order, one can define subsequently contract internal edges of Γ, and obtain a map

$$\prod_{v\in V_{int}(\Gamma)} \mathrm{Hom}((X_e)_{e\in Star_{in}(v)}, X_{Star_{out}(v)}) \longrightarrow Hom((X_e)_{e\in E_{in}(\Gamma)},(X_e)_{E_{out}(\Gamma)})$$

The axiom is that this map should not depend on numerations, and on the order in which we perform compositions and tensor products.

(4) for any finite collection of objects $(X_i)_{i\in I}$ there exists an object $Y \in Ob(C)$ and morphisms $f \in \mathrm{Hom}((X_i)_{i\in I}, Y), g \in \mathrm{Hom}(Y,(X_i)_{i\in I})$ such that composition $f \circ g$ is equal to id_Y, and composition $g \circ f$ is equal to the tensor product of $(id_{X_i})_{i\in I}$.

The structure of a category is given by polymorphisms betwen one-element families. The object Y in the last axiom is defined uniquely up to a canonical isomorphism. We denote it by $\otimes_{i\in I} X_i$. The identity object $\mathbf{1}_{\mathcal{C}}$ is defined as the tensor product of the empty family of objects.

If one omits the last axiom, one get a weaker structure which arises if one pick an *arbitrary* subclass (and denote it by $Ob(\mathcal{C})$ in $Ob(\tilde{\mathcal{C}})$ where $\tilde{\mathcal{C}}$ is a symmetric monoidal category.

9.0.2. *Examples.* Here we show few examples of “non-linear” symmetric monoidal categories:

Example 9.0.2. Let $\mathcal{C} = Sets$ be the category of sets, the tensor product is defined as $X \otimes Y := X \times Y$, the unit object is the one-element set. Morphisms of associativity, commutativity and unit are obvious. Analogous definitions can be done in arbitrary category with finite products, e.g. for groups, vector spaces, topological spaces, etc.

EXAMPLE 9.0.3. Again in the category *Sets* , we define the tensor product as $X \otimes Y := X \sqcup Y$, the unit object is the empty set. Analogous definition works in any category with finite sums.

EXAMPLE 9.0.4. $\mathcal{C}$ is the category of A-modules where A is a commutative associative unital ring. The tensor product is the usual tensor product over A.

9.0.3. *Monoidal categories.* This is a weakened notion relative to symmetric monoidal categories. One should remove permutations of indices from the data, and consider graphs Γ endowed with complete orderings on the sets $Star_{in}(v)$, $Star_{out}(v)$ for all vertices. The tensor product $\otimes_{i\in I} X_i$ is defined in monoidal category if the labeling set I is totally ordered.

EXAMPLE 9.0.5. Let $\mathcal{C}$ be a small category (i.e. the class of objects is in fact a set). Then the category of functors $Funct(\mathcal{C}, \mathcal{C})$ is a monoidal category. Morphisms are natural transformations of functors, monoidal structure is given by the composition, the unit object is the identity functor $Id_{\mathcal{C}}$.

EXAMPLE 9.0.6. Let A be an associative unital algebra. The category $\mathcal{C}$ of bimodules over A, (i.e. $A \otimes A^{op}$-modules is a monoidal category.

9.0.4. *Braided monoidal categories.* This is an intermediate notion between monoidal and symmetric monoidal categories. The tensor product $\otimes_{i\in I} X_i$ is defined if the labeling set I is a subset of $\mathbf{R^2}$. Let I_1 and I_2 be two n-element subsets of $\mathbf{R^2}$. Any homotopy class of paths between I_1 and I_2 in the space $\{n -$ element subsets of $\mathbf{R^2}\}$ should give a canonical isomorphism between corresponding tensor products. In particular, for any object X on its n-th tensor power corresponding to the subset $\{(1, 0, (2, 0), \dots, (n, 0\} \subset \mathbf{R^2}$ acts the braid group B_n (nad not Σ_n as for symmetric monoidal categories). We will not discuss here in details the notion of a braided monoidal category because it will be not used further.

9.0.5. *Functors between symmetric monoidal categories.*

DEFINITION 9.0.7. Let $\mathcal{C}$ and $\mathcal{C}'$ be two symmetric monoidal categories. A *symmetric monoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ consists of a map $F : Ob(\mathcal{C}) \longrightarrow Ob(\mathcal{C}')$, and maps

$$\mathrm{Hom}_{\mathcal{C}}((X_i)_{i\in I}, (Y_j)_{j\in J})) \longrightarrow \mathrm{Hom}_{\mathcal{C}'}(((F(X_i))_{i\in I}, (F(Y_j))_{j\in J}))$$

such that permutations, identity morphisms, tesnor products and compositions in $\mathcal{C}$ go to analogous operations in $\mathcal{C}'$.

One can also define the notion of a natural transformation between two symmetric monoidal functors. Also, one can drop the commutativity and speak about monoidal functors between monoidal categories.

EXERCISE 9.0.8. Fill the detailes in definitions and prove if $\mathcal{C}$ is a monoidal category then the category of monoidal endofunctors of $\mathcal{C}$ carries a natural structure of a braided monoidal category.

9.0.6. *Enrichment by symmetric monoidal categories.* In many definition in category theory one can replace sets by objects of symmetric monoidal categories. For example, see what happens if one tries to replace *sets of morphisms* in categories by something else:

DEFINITION 9.0.9. Let $\mathcal{C}$ be a symmetric monoidal category. A *$\mathcal{C}$-enriched category* B consists of a class of objects $Ob(B)$, an object $\underline{\mathrm{Hom}}(X,Y) \in Ob(\mathcal{C})$ for any $X, Y \in Ob(B)$, the unit $id_X : \mathbf{1}_{\mathcal{C}} \longrightarrow \underline{\mathrm{Hom}}(X,X)$ (a morphism in $\mathcal{C}$) for any $X \in \mathcal{C}$ and the composition $comp_{X,Y,Z} : \underline{\mathrm{Hom}}(X,Y) \otimes \underline{\mathrm{Hom}}(Y,Z) \longrightarrow \underline{\mathrm{Hom}}(X,Z)$ (another morphism in $\mathcal{C}$) satisfying obvious analogs of the usual axioms.

Any category is automatically $Sets$-enriched with the monoidal structure on $Sets$ given by the cartesian product. Conversely, for any $\mathcal{C}$-enriched category one can define a usual category structure, with the same class of objects and with the sets of morphisms given by

$$Hom(X,Y) := \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \underline{\mathrm{Hom}}(X,Y))$$

A k-linear category is the same as $Vect_k$-enriched category. Analogously, one can defined categories whose morphism sets are topological spaces, simplicial stes, etc.

The same can be done with definition of a symmetric monoidal category. In particular, tensor k-linear categories are $Vect_k$-enriched symmetric monoidal categories (satisfying some additional properties such as to be abelian, etc.). Any symmetric monoidal category $\mathcal{C}$ with $\underline{\mathrm{Hom}}$-s can be enriched by itself.

CHAPTER 3

Differential-graded manifolds

1. Formal manifolds in tensor categories

1.1. Formal manifolds and coalgebras. Let k be a field of characteristic zero, $\mathcal{C}$ be a k-linear abelian tensor category, which we will assume to be either $Vect_k$, $Super_k$ or $Vect_k^{\mathbf{Z}}$. Then for any object $V \in Ob(\mathcal{C})$ and $n \geq 0$ we have a natural action of the symmetric group S_n on $V^{\otimes n}$ (by definition S_0 and S_1 are trivial groups). In particular, we can define the vector space of symmetric tensors $S^n(V) = (V^{\otimes n})^{S_n}$. More generally, we can do linear algebra in $\mathcal{C}$. In particular we have the notions of associative, commutative or Lie algebra in the category $\mathcal{C}$ (we will explain later how to define more general notion of an algebra over an operad). One has also the notion of coalgebra in $\mathcal{C}$. Coalgebras with S_2-invariant coproducts are called *cocomutative.* In this chapter we are going to consider cocomutative coalgebras only.

Let A be a cofree coassociative cocommutative coalgebra in $\mathcal{C}$. We assume that A does not have a counit. Then by definition, there exists $V \in Ob(\mathcal{C})$ such that

$$A \simeq C(V) := \oplus_{n\geq 1} S^n(V),$$

The coalgebra structure on $C(V)$ is given by the coproduct $\Delta : C(V) \to C(V) \otimes C(V)$ such that $\Delta(v) = 0$, for $v \in V$, and

$$\Delta(v_1 \otimes ... \otimes v_n) = \sum_{\sigma \in S_n} \sum_{1 \leq i \leq n-1} \frac{1}{n!} (v_{\sigma(1)} \otimes ... \otimes v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \otimes ... \otimes v_{\sigma(n)}).$$

Definition 1.1.1. We say that A is conilpotent if for some $n \geq 2$ the iterated coproduct $\Delta^{(n)} : A \to A^{\otimes n}$ is equal to zero. We say that A is locally conilpotent if for any $a \in A$ there exists $n \geq 1$ such that $\Delta^{(n)}(a) = 0$.

Recall that the iterated coproducts are defined by the formulas $\Delta^{(2)} = \Delta, \Delta^{(n)} = (\Delta \otimes id^{\otimes n})\Delta^{(n-1)}$.

The union of locally conilpotent coalgebras is a locally conilpotent coalgebra. For coalgebras of finite length (i.e. they are Artin objects in $\mathcal{C}$) local conilpotency and conilpotency coincide. From now on, unless we say otherwise, all our coalgebras will be locally conilpotent.

Exercise 1.1.2. a) Any locally conilpotent coalgebra in the category of vector spaces is a union of finite-dimensional conilpotent coalgebras.

b) Let $V \mapsto Coalg_+(V) := C(V)$ be the functor which assigns to a graded vector space V the cofree coassociative cocommutative coalgebra without counit generated by V. Then $Coalg_+$ is the left adjoint to the forgetful functor from the category of conilpotent cocommutative coalgebras without counit to $Vect_k^{\mathbf{Z}}$.

Clearly, coalgebras without counit form a category, which we will denote by $Coalg_{\mathcal{C}}$. Counital coalgebras form a category which will be denoted by $(Coalg_1)_{\mathcal{C}}$. We will usually skip $\mathcal{C}$ from the notation.

Definition 1.1.3. Formal pointed manifold in the category $\mathcal{C}$ is an object of the category of cofree conilpotent coalgebras without counit.

An isomorphism of the coalgebra with $C(V)$ is not a part of the data. We will denote a formal pointed manifold by (X, x_0) or by (X, pt). The corresponding coalgebra $C = C_X$ should be thought of as the coalgebra of distributions on X supported at the marked point.

Definition 1.1.4. Formal pointed manifold in the tensor category $Vect_k^{\mathbf{Z}}$ of $\mathbf{Z}$-graded k-vector spaces is called formal pointed graded (or $\mathbf{Z}$-graded) manifold.

The case of formal pointed graded manifolds will be especially interesting for us. At the same time, we would like to stress that many facts remain true for general abelian tensor categories.

Formal pointed manifolds form a category. Morphisms of formal pointed manifolds correspond to homomorphisms of coalgebras. Geometrically morphisms can be thought of as formal maps preserving based points.

Let us assume that $\mathcal{C}$ is a rigid category (which is true for all three examples we have in mind). Then for any object V we can canonically define the dual object V^*. In particular, we have the dual object $C(V)^* := \prod_{n\geq 1}(S^n(V))^*$. If B is a cofree locally conilpotent coalgebra, then B^* is a projective limit of finite-dimensional nilpotent algebras. In the case $B = C_X$ call it *the algebra of functions* on the formal neighborhood of the marked point x_0 vanishing at x_0.

Example 1.1.5. Let V be a vector space. Then the dual algebra $(C(V))^* = \prod_{n\geq 1}(S^n(V))^*$ is isomorphic to the algebra of formal power series $k[[t_i]]_{i\in I}$ vanishing at zero. Here the cardinality of I is equal to the dimension of V. This example explains our terminology. The coalgebra $C(V)$ corresponds to the formal affine space with a marked base point: $X = (V_{form}, 0)$.

Recall that in general we do not fix an isomorphism between A and $C(V)$. In geometric language of the previous example this means that we do not fix affine coordinates on $(V_{form}, 0)$. Fixing affine coordinates is equivalent to a choice of an isomorphism of coalgebras $A \simeq C(V)$.

If $A \simeq C(V)$ is a cofree coalgebra without counit, then one can canonically construct a cofree coalgebra *with* the counit ϵ. As a vector space it is given by $\widehat{A} := A \oplus k$. The coalgebra structure on $\widehat{A}$ is uniquely defined by the formulas $\Delta(1) = 1 \otimes 1$, $\Delta(v) = v \otimes 1 + 1 \otimes v$, where $v \in V$. The counit is defined by $\epsilon(1) = 1, \epsilon(v) = 0$. The counital free coalgebras correspond to the objects which we will call *formal manifolds* in $\mathcal{C}$ (or, simply, formal manifolds, if it will not lead to a confusion). We use the same terminology (algebra of functions, coalgebra of distributions, etc.) as before, omitting the conditions at the marked point.

We prefer to work with coalgebras rather than algebras. It simplifies the treatment of the case $dim(V) = \infty$. The definition of $C(V)$ is pure algebraic (it uses direct sums), so it can be dualized. If we start with the algebra of formal power series then dualization is a more delicate issue. On the other hand, as long as we work with cocommutative counital coalgebras, we are close to the finite-dimensional situation. Indeed, the following proposition holds.

Proposition 1.1.6. *Any cocommutative counital coalgebra A in the category of k-vector spaces is a union of finite dimensional subcoalgebras. If A is locally conilpotent then the finite-dimensional subcoalgebras can be chosen conilpotent.*

Proof. Second part of the Proposition coinsides with the part a) of the Exercise 3.2. Let us proof the first part. We have: $\Delta(a) = \sum_i x_i \otimes y_i$. The linear span A_a of the x_i (which equals by cocommutativity to the linear span of y_i) is finite-dimensional. We can choose the vectors x_i to be linearly independent, and the vectors y_i to be linearly independent. Let $\varepsilon : A \to k$ be the counit. Then $(\varepsilon \otimes id)\Delta(a) = a$ and hence $a = \sum_i \varepsilon(x_i)y_i \in A_a$. Let us prove that A_a is a subscoalgebra. We need to prove that $\Delta(A_a) \subset A_a \otimes A_a$. Coassociativity condition implies that $\sum_i \Delta(x_i) \otimes y_i = \sum_i x_i \otimes \Delta(y_i)$. We can write $\Delta(y_i) = \sum_j y^1_{ij} \otimes y^2_{ij}$ where vectors y^1_{ij} (resp. y^2_{ij}) are linearly independent. Any vector $v \in A^{\otimes 3}$ admits a unique presentation in the form $v = \sum_i m_i \otimes n_i \otimes l_i$ with each group m_i, n_i and l_i to be linearly independent. Hence we have $Span\{y^1_{ij}\} = Span\{y^2_{ij}\} = Span\{y_i\}$ (first equality follows from the cocommutativity condition). Therefore A_a is a subcoalgebra of A. The sum of such subcoalgebras is clearly a subcoalgebra. Obviously we can represent A as a union of such sums. ■

REMARK 1.1.7. The Proposition holds for non-counital algebras. Moreover, later we will prove it for non-cocommutative coalgebras as well.

Using last Proposition we can offer a more conceptual point of view on formal pointed manifolds. Indeed, we see that a cocommutative non-unital coalgebra B gives rise to an ind-object in the category of finite-dimensional cocommutative non-counital coalgebras in $Vect_k$. If we write $B = \varinjlim_I B_i$, where B_i are finite-dimensional cocommutative coalgebras, then we have a covariant functor $F_B : Artin_k \to Sets$ such that $F_B(R) = \varinjlim_I Hom_{Coalg_k}(B_i^*, R^*)$. The functor F_B commutes with finite projective limits. The converse is also true. We will formulate below the result for counital cocommutative coalgebras, skipping the proof. More general result for arbitrary coalgebras will be proved in Chapter 6.

PROPOSITION 1.1.8. *Let $F : Artin_k \to Sets$ be a covariant functor commuting with finite projective limits. Then there exists a counital coalgebra B such that F is isomorphic to the functor $R \mapsto Hom_{Coalg_k}(R^*, B)$.*

It is easy to show that the category of counital coalgebras is equivalent to the category of functors described in the above Proposition. We see that counital coalgebras in $\mathcal{C}$ give rise to ind-schemes in this tensor category (see Appendix for the terminology of ind-schemes). We are going to call them *small schemes*. In Chapter 6 we are going to generalize these considerations to the case of not necessarily cocommutative coalgebras. It will be achieved by considering functors to *Sets* from the category of Artin algebras, which are not-necessarily commutative. In particular, we are going to discuss the notion of smoothness. All the proofs from Chapter 6, Section 2.3 admit straightforward versions IN the cocommutative case, so we omit them here.

DEFINITION 1.1.9. A cocommutative coalgebra B (or the corresponding small scheme) is called smooth, if for any morphism $A_1 \to A_2$ of finite-dimensional cocommutative coalgebras the corresponding map of sets $Hom_{Coalg}(A_2, B) \to Hom_{Coalg}(A_1, B)$ is surjective.

Let A be a locally conilpotent cocommutative coalgebra without counit. It is natural to ask under which conditions it defines a formal pointed manifold. We present below without a proof the answer for the category $Vect_k$ of vector spaces.

THEOREM 1.1.10. *Let $F_n(A)$ be a filtration of A defined by the kernels of the iterated coproducts $\Delta^{(n)} : A \to A^{\otimes n}$. Suppose that $A = \varinjlim F_n(A)$. Then A is cofree if $gr(A) = \oplus_{n \geq 0} F_n(A)/F_{n+1}(A)$ is cofree (i.e. $gr(A) \simeq \oplus_{n\geq 0} S^n(F_1(A))$).*

In this case A is smooth. Conversely, if A is smooth then it is isomorphic to the coalgebra $C(V)$ for some vector space V.

Dual result in the category of vector spaces is the Serre's theorem (criterium of smoothness of a formal scheme).

1.2. Vector fields, tangent spaces. Let us return to the case of an arbitrary k-linear tensor category. One can translate from algebraic to geometric language and back many structures of formal geometry.

DEFINITION 1.2.1. a) Vector field on a formal pointed manifold, which corresponds to a non-counital coalgebra A, is given by a derivation of the corresponding counital coalgebra $\widehat{A}$.

b) Vector field is called vanishing at the based point if it is given by a derivation of A (i.e. it is a derivation of $\widehat{A}$ preserving A).

c) Tangent space $T_{pt}(X)$ to a formal pointed manifold (X, pt) is the object $Ker(\Delta)$, where $\Delta : A \to A^{\otimes 2}$ is the coproduct.

Let (X_i, pt_i) be formal pointed manifolds. We assume that we have chosen affine coordinates. This means a choice of isomorphisms $C_i \simeq C(V_i), i = 1, 2$ of the corresponding cofree coalgebras. Let $f : (X_1, pt_1) \to (X_2, pt_2)$ be a morphism of formal pointed manifolds. By definition it corresponds to the homomorphism of cofree cocommutative coalgebras $\mathcal{F} : C_1 \to C_2$. Because of the universality property, it is uniquely determined by the composition $pr_2 \circ \mathcal{F}$ where $pr_2 : C_2 \to Ker(\Delta_{C_2})$ is the projection. Notice that the projection depends of a choice of the isomorphism $C_2 \simeq C(V_2)$ (although the kernel of the coproduct doesn't). Restricting $pr_2 \circ \mathcal{F}$ to $Ker(\Delta_{C_1})$ we obtain a linear map between the tangent spaces $T(f) : T_{pt_1}(X_1) \to T_{pt_2}(X_2)$.

We will denote $T(f)$ also by f_1 and treat it as the first Taylor coefficient of f at the based point pt_1.

The homomorhism of coalgebras $\mathcal{F} : C(V_1) \to C(V_2)$ is uniquely determined by a sequence of linear maps $\mathcal{F}_n : S^n(V_1) \to V_2$, such that $\mathcal{F}_n = pr_2 \circ \mathcal{F}|_{S^n(V)}$. We will say that the sequence $(\mathcal{F}_n)_{n\geq 1}$ determines the Taylor decomposition $f = \sum_{n \geq 1} f_n$ of the morphism f. We are not going to distinguish between $\mathcal{F}_n$ and f_n in the future, calling either of them the Taylor coefficients of f.

It is easy to see that f_n can be identified with the linear map $\partial^n f(v_1 \cdot ... \cdot v_n) = \frac{\partial^n}{\partial x_1 ... \partial x_n}{}_{|x_1 = ... x_n = 0} f(x_1 v_1 + ... + x_n v_n)$ where x_i are affine coordinates in V_1. To do this one has to interpret $S^n(V_1)$ as the quotient space of $V_1^{\otimes n}$ rather than a subspace of invariants.

1.3. Inverse function theorem.

THEOREM 1.3.1. *Let (X_1, pt_1) and (X_2, pt_2) be formal pointed manifolds, and $C_i, i = 1, 2$ the corresponding cofree coalgebras. Then a morphism $f : (X_1, pt_1) \to (X_2, pt_2)$ is an isomorphism if and only if the induced linear map of tangent spaces $f_1 = T(f) : T_{pt_1}(X_1) \to T_{pt_2}(X_2)$ is an isomorphism.*

Proof. Clearly C_1 and C_2 are filtered, where the filtrations are defined by the kernels of the coproducts: $F_n(C_i) = \mathrm{Ker}((\Delta \otimes id \otimes ... id)...(\Delta \otimes id)\Delta))((n+1)$ times,

$n \geq 0, i = 1, 2)$. Morphism f is compatible with the filtrations. Using induction by n we see that f is an isomorphism. ∎

The inverse mapping theorem admits a generalization called *implicit mapping theorem.* We are going to formulate it without proof. The proof is left as an exercise to the reader.

Before stating the theorem, we remark that there are finite products in the category of formal pointed manifolds. They correspond to the tensor products of the corresponding coalgebras. Same is true for non-pointed formal manifolds.

THEOREM 1.3.2. *Let $f : (X_1, pt_1) \to (X_2, pt_2)$ be a morphism of formal pointed manifolds such that the corresponding tangent map $f_1 : T_{pt_1}(X_1) \to T_{pt_2}(X_2)$ is an epimorphism. Then there exists a formal pointed manifold (Y, pt_Y) such that $(X_1, pt_1) \simeq (X_2, pt_2) \times (Y, pt_Y)$, and under this isomorphism f becomes the natural projection.*

If f_1 is a monomorphism, then there exists (Y, pt_Y) and an isomorphism $(X_2, pt_2) \to (X_1, pt_1) \times (Y, pt_Y)$, such that under this isomorphism f becomes the natural embedding $(X_1, pt_1) \to (X_1, pt_1) \times (pt_Y, pt_Y)$.

If S is a k-scheme then one can speak about a *family of formal pointed manifolds over S.* Namely, the family is given by a quasi-coherent sheaf F of cocommutative coalgebras without counit, which is locally finitely generated, and every fiber of F is a smooth coalgebra.

Similarly one can define a family of formal pointed manifolds over a base S which is itself a formal manifold.

2. Formal pointed dg-manifolds

2.1. Main definition.

DEFINITION 2.1.1. A formal pointed differential-graded manifold (dg-manifold for short) over k is a pair (M, Q) consisiting of a formal pointed $\mathbf{Z}$-graded manifold M and a vector field Q on M of degree $+1$ such that $[Q, Q] = 0$.

As before, we will often skip $\mathbf{Z}$ from the notation. We will also often skip Q, thus denoting by M the formal pointed dg-manifold $((M, pt), Q)$. Unless we say otherwise, we assume that the vector field Q vanishes at the marked point pt.

A formal pointed graded manifold is modelled by a cofree cocommutative coalgebra $C(V)$ generated by a graded vector space V. A formal pointed dg-manifold is modelled by a pair $(C(V), Q)$, where $C(V)$ is as above and Q is a derivation of the coalgebra $C(V)$ of degree $+1$, such that $Q^2 = 0$. Then a morphism of formal pointed dg-manifolds is defined as a homomorphism of the corresponding coalgebras which commutes with the differentials. In this way we obtain a category of formal pointed dg-manifolds. It is a symmetric monoidal category with the tensor product given by the tensor product of differential coalgebras.

Let $M = (C(V), Q)$ be a formal pointed dg-manifold. Then V carries a structure of L_∞*-algebra*, which is a generalization of that of a Lie algebra. For that reason dg-manifolds are sometimes called L_∞*-manifolds.* We are going to discuss L_∞-algebras below.

2.2. Remark about non-formal dg-manifolds. Occasionally we will be using graded manifolds (and dg-manifolds) in the non-formal set-up. We can define graded (and differential-graded) manifolds in the following categories:

a) category of smooth manifolds;
b) category of algebraic manifolds over a field of characteristic zero.

DEFINITION 2.2.1. A graded smooth manifold is an S^1-equivariant smooth supermanifold such that $-1 \in S^1$ acts as the canonical involution (the latter changes the parity on the supermanifold).

A smooth dg-manifold is a graded manifold which carries a vector field Q of degree $+1$ such that $[Q,Q]=0$.

REMARK 2.2.2. Replacing smooth supermanifolds by algebraic supermanifolds, and the group S^1 by the multiplicative group G_m one gets the definitions of a graded algebraic manifold and an algebraic dg-manifold. We will be using these definitions later in the book.

2.3. L_∞-algebras. Let V be a $\mathbf{Z}$-graded vector space. As before, we denote by $C(V)$ the cofree cocommutative coassociative coalgebra without counit generated by V.

DEFINITION 2.3.1. An L_∞-algebra is a pair (V,Q) where V is a $\mathbf{Z}$-graded vector space and Q is a differential on the graded coalgebra $C(V[1])$.

Thus we see that an L_∞-algebra (V,Q) gives rise to a formal pointed dg-manifold $((V[1]_{formal},0),Q)$. One can say that a formal pointed dg-manifold is locally modelled by an L_∞-algebra.

It is useful to develop both algebraic and geometric languages while speaking about formal pointed dg-manifolds. This subsection is devoted to the algebraic one.

The derivation Q is determined by its restriction to cogenerators, i.e. by the composition

$$\oplus_{n\geq 1} S^n(V[1]) = C(V[1]) \xrightarrow{Q} C(V)[2] \xrightarrow{\text{projection}} V[2].$$

This gives rise to a collection of morphisms of graded vector spaces

$$Q_n : S^n(V[1]) \to V[2]$$

satisfying an infinite system of quadratic equations (all encoded in the equation $Q^2=0$).

Since $S^n(V[1]) \simeq \wedge^n(V)[n]$ (prove it) the maps Q_n give rise to a collection of "higher brackets"

$$[\ ,\ \dots\ ,\ ]_n : \wedge^n(V) \to V[2-n],$$

for $n=1,2,...$

Slightly abusing the notation we will often denote these brackets by the same letters Q_n.

The condition $Q^2=0$ gives rise to a sequence of the following identities (they hold for every $n\geq 1$ and homogeneous $v_1,...,v_n$):

$$\sum_{\sigma\in S_n}\ \sum_{k,l\geq 1, k+l=n+1} \pm[[v_{\sigma(1)},...,v_{\sigma(k)}]_k,....,v_{\sigma(n)}]_l = 0,$$

where S_n is the symmetric group.

Let us consider first few identities:

a) $n=1$ equation is just $Q_1^2(v) = [[v]_1]_1 = 0$. Hence $Q_1 = [\,]_1 : V \to V[1]$ defines a differential on V.

b) $n = 2$ equation means that $Q_2 = [\,,\,]_2 : \wedge^2(V) \to V$ is a homomorphism of complexes with the differentials induced by Q_1.

c) $n = 3$ equation means that $[\,,\,]_2$ satisfies Jacobi identity up to homotopy given by $Q_3 = [\,,\,,\,]_3$.

As a corollary we have the following result.

PROPOSITION 2.3.2. *Let $H^*(V)$ be the cohomology of V with respect to Q_1. Then the bracket $Q_2 = [\,,\,]_2$ defines a structure of $\mathbf{Z}$-graded Lie algebra on $H^*(V)$.*

EXERCISE 2.3.3. Prove that DGLA is an L_∞-algebra such that $[...]_k = 0$ for $k = 3, 4, ...$

REMARK 2.3.4. Sometimes L_∞-algebras are called *strong homotopy Lie algebras* (SHLA) or simply homotopy Lie algebras.

2.4. Morphisms of L_∞-algebras. By definition, a morphism of L_∞-algebras is a morphism of the corresponding differential-graded coalgebras $f : C(V_1)[1] \to C(V_2)[1]$.

We know that free commutative algebras are defined by the functorial property $Hom_{\mathrm{Alg}}(\mathrm{Comm}(V), B) = \mathrm{Hom}(V, B)$.

Analogously, cofree cocommutative coalgebras are defined by the property $Hom_{\mathrm{Coalg}}(B, \mathrm{Cofree}(V)) = \mathrm{Hom}(B, V)$ which holds for every cocommutative coalgebra B without counit.

Thus a morphism of cofree coalgebras corresponding to two L_∞-algebras is an infinite collection of maps

$$f_n : \wedge^n(V_1) \to V_2[1 - n].$$

Compatibility with Q turns into a sequence of equations.

EXERCISE 2.4.1. Write down these equations. Show that f_1 is a morphism of complexes, which is compatible with $[\,,\,]_2$ up to homotopy.

Notice that for DGLAs V_1, V_2 there are more morphisms in the category of L_∞-algebras than in the category of DGLAs. This is one of the reasons why the former category is better adopted for the purposes of the homotopy theory.

2.5. Pre-L_∞-morphisms. This subsection serves a technical purpose. It contains formulas which will be used later.

Let g_1 and g_2 be two graded vector spaces.

DEFINITION 2.5.1. A pre-L_∞-morphism F from g_1 to g_2 is a map of formal pointed manifolds

$$F : ((g_1[1])_{formal}, 0) \to ((g_2[1])_{formal}, 0)$$

The map F is defined by its Taylor coefficients which are, by definition, linear maps $\partial^n F$ of graded vector spaces:

$$\partial^1 F : g_1 \to g_2$$

$$\partial^2 F : \wedge^2(g_1) \to g_2[-1]$$

$$\partial^3 F : \wedge^3(g_1) \to g_2[-2]$$

$$\cdots$$

Here we use again the natural isomorphism $S^n(g_1[1]) \simeq \big(\wedge^n(g_1)\big)[n]$. Equivalently, we have a collection of linear maps between vector spaces

$$F_{(k_1,\dots,k_n)} : g_1^{k_1} \otimes \dots \otimes g_1^{k_n} \to g_2^{k_1+\dots+k_n+(1-n)}$$

with the symmetry property

$$F_{(k_1,\dots,k_n)}(\gamma_1 \otimes \dots \otimes \gamma_n) = -(-1)^{k_i k_{i+1}} F_{(k_1,\dots,k_{i+1},k_i,\dots,k_n)}(\gamma_1 \otimes \dots \otimes \gamma_{i+1} \otimes \gamma_i \otimes \dots \otimes \gamma_n) \ .$$

Here g_i^n denotes the nth graded component of $g_i, i = 1, 2$.

One can write (slightly abusing the notation)

$$\partial^n F(\gamma_1 \wedge \dots \wedge \gamma_n) = F_{(k_1,\dots,k_n)}(\gamma_1 \otimes \dots \otimes \gamma_n)$$

for $\gamma_i \in g_1^{k_i},\ i = 1, \dots, n$.

In the sequel we will denote $\partial^n F$ simply by F_n.

2.6. L_∞-algebras and formal pointed dg-manifolds. Recall that an L_∞-algebra (g, Q) gives rise to a formal pointed dg-manifold $((g[1]_{formal}, 0), Q)$. This means that an L_∞-algebra is the same as a formal pointed dg-manifold with an affine structure at the marked point (i.e. a choice of an isomorphism of the coalgebra of distributions with $C(V)$).

Let g_1 and g_2 be L_∞-algebras. Then an L_∞-morphism between them is a pre-L_∞-morphism compatible with the differentials. Equivalently, it is a morphism of formal pointed manifolds $(g_1[1]_{formal}, 0) \to (g_2[1]_{formal}, 0)$ commuting with the corresponding odd vector fields. We can also say that it is a morphism in the category of formal pointed dg-manifolds.

In the case of differential-graded Lie algebras a pre-L_∞-morphism F is an L_∞-morphism iff its Taylor coefficients satisfy the following equation for any $n = 1, 2 \dots$ and homogeneous elements $\gamma_i \in g_1$:

$$dF_n(\gamma_1 \wedge \gamma_2 \wedge \dots \wedge \gamma_n) - \sum_{i=1}^{n} \pm F_n(\gamma_1 \wedge \dots \wedge d\gamma_i \wedge \dots \wedge \gamma_n) =$$

$$= \frac{1}{2} \sum_{k,l \geq 1,\ k+l=n} \frac{1}{k!l!} \sum_{\sigma \in S_n} \pm [F_k(\gamma_{\sigma_1} \wedge \dots \wedge \gamma_{\sigma_k}), F_l(\gamma_{\sigma_{k+1}} \wedge \dots \wedge \gamma_{\sigma_n})] +$$

$$\sum_{i<j} \pm F_{n-1}([\gamma_i, \gamma_j] \wedge \gamma_1 \wedge \dots \wedge \gamma_n) \ .$$

Here are first two equations:

$$dF_1(\gamma_1) = F_1(d\gamma_1)\,,$$

$$dF_2(\gamma_1 \wedge \gamma_2) - F_2(d\gamma_1 \wedge \gamma_2) - (-1)^{\overline{\gamma_1}} F_2(\gamma_1 \wedge d\gamma_2) = F_1([\gamma_1, \gamma_2]) - [F_1(\gamma_1), F_1(\gamma_2)] \ .$$

We see that F_1 is a morphism of complexes. The same is true for arbitrary L_∞-algebras. The graded space g for an L_∞-algebra (g, Q) can be considered as the tensor product of $k[-1]$ with the tangent space to the corresponding formal pointed manifold at the base point. The differential Q_1 on g arises from the action of Q on the manifold. In other words, the tangent space at the base point is a complex of vector spaces with the differential given by the first Taylor coefficient of the odd vector field Q.

Let us assume that g_1 and g_2 are differential-graded Lie algebras, and F is an L_∞-morphism from g_1 to g_2. Any solution $\gamma \in g_1^1 \otimes m$ to the Maurer-Cartan equation where m is a commutative nilpotent non-unital algebra, produces a solution to the Maurer-Cartan equation in $g_2^1 \otimes m$:

$$d\gamma + \frac{1}{2}[\gamma,\gamma] = 0 \Longrightarrow d\widetilde{\gamma} + \frac{1}{2}[\widetilde{\gamma},\widetilde{\gamma}] = 0 \quad \text{where} \quad \widetilde{\gamma} = \sum_{n=1}^{\infty} \frac{1}{n!} F_n(\gamma \wedge \cdots \wedge \gamma) \in g_2^1 \otimes m \ .$$

The same formula is applicable to solutions to the Maurer-Cartan equation depending formally on a parameter h:

$$\gamma(h) = \gamma_1 h + \gamma_2 h^2 + \cdots \in g_1^1[[h]], \;\; d\gamma(h) + \frac{1}{2}[\gamma(h),\gamma(h)] = 0$$

This implies the following equation:

$$d\widetilde{\gamma(h)} + \frac{1}{2}[\widetilde{\gamma(h)}, \widetilde{\gamma(h)}] = 0 \ .$$

REMARK 2.6.1. In order to understand conceptually the last implication, we need to use the notion of a formal dg-manifold *without* the base point. Then one observes that the Maurer-Cartan equation for any differential-graded Lie algebra g is the equation for the subscheme of zeroes of Q in formal manifold $g[1]_{formal}$. Clearly L_∞-morphism $f : (M_1,Q_1) \to (M_2,Q_2)$ maps the subscheme of zeros of Q_1 to the one of Q_2 (because f commutes with $Q_i, i = 1,2$). Using this observation we will see later that the L_∞-morphism f induces a natural transformations of deformation functors defined by (M_1,Q_1) and (M_2,Q_2) respectively.

2.7. Tangent complex. We have already introduced the notion of the tangent space to a formal pointed manifold. Let us recall it here. The dual space to a cofree coalgebra $C(V) = \oplus_{n\geq 1} S^n(V)$ is an algebra of formal power series $C^* = \prod_{n\geq 1}(S^n(V))^*$ (without the unit). Adding the unit we obtain the algebra of *formal functions* on a formal pointed manifold (maybe, infinite-dimensional) with the marked point 0. Algebraically a "choice of affine coordinates" corresponds to the identification of the formal scheme $\mathrm{Spf}(C^*)$ with the formal neighborhood of zero. The (graded) tangent space is $T_0(C) := \mathrm{Ker}(\Delta : C \to C \otimes C)$. .

Recall that we have the notion of formal graded manifolds without marked point. Such manifolds form a category dual to the category of cocommutative cofree coalgebras which are isomorphic to $\oplus_{n\geq 0} S^n(V)$. The definition of formal dg-manifold is also clear. Suppose we have fixed a closed k-point x_0 of a formal dg-manifold (M,Q). The odd vector field Q can have non-trivial component $Q_0 = Q(x_0)$. If $Q_0 = 0$ we can factorize the corresponding coalgebra by the zeroth component and obtain a new coalgebra which represents a formal pointed graded manifold. In the non-vanishing case the following theorem holds.

THEOREM 2.7.1. *Let (M,Q) be a formal dg-manifold and pt a closed k-point. Suppose that $Q(pt) \neq 0$. Then by a formal change of coordinates preserving pt one can make the vector field Q equivalent to a vector field with constant coefficients. Equivalently, in some coordinates (x_i) we have: $Q = \partial/\partial x_1$.*

Proof. Exercise. ■

This result is similar to the one in the theory of ordinary differential equations: a vector field is locally equivalent to a constant one near a point where it is non-zero. The classification of critical points is hard in both cases. Since we are interested in L_∞-algebras, we will almost exclusively consider the case of formal *pointed* manifolds unless we say otherwise.

Let us recall that in the category of formal pointed manifolds we have the inverse image theorem. It says that the formal morphism is invertible iff its first Taylor coefficient is invertible. We would like to generalize the theorem to the category of formal pointed dg-manifolds.

Suppose that f is the homomorphism of coalgebras corresponding to a morphism of formal pointed dg-manifolds. Since f commutes with Q, we see that the tangent map f_1 commutes with its first Taylor coefficient Q_1. If C is the coalgebra corresponding to a formal pointed dg-manifold $((M, pt), Q)$ then on $T_{pt}(M) = Ker(\Delta)$ arises a differential which is the first Taylor coefficient Q_1 of Q.

DEFINITION 2.7.2. The pair $(T_{pt}(M), Q_1)$ considered as a complex of vector spaces is called the tangent complex of M at the base point.

DEFINITION 2.7.3. Two formal pointed dg-manifolds (resp. L_∞-algebras) are called quasi-isomorphic if the corresponding differential-graded coalgebras are quasi-isomorphic.

Let (M, Q) and (M', Q') be formal pointed dg-manifolds.

DEFINITION 2.7.4. Tangent quasi-isomorphism (t-qis for short) from (M, Q) to (M', Q') is a morphism of these formal pointed dg-manifolds such that the corresponding morphism of tangent complexes is a quasi-isomorphism.

Suppose that our formal pointed dg-manifolds correspond to L_∞-algebras $g_i, i = 1, 2$. Then a *quasi-isomorphism* of L_∞-algebras $g_1 \to g_2$ is defined as the t-qis of the corresponding formal pointed dg-manifolds $g_1[1]_{formal} \to g_2[1]_{formal}$. Equivalently, it is a homomorphism of dg-coalgebras $f : C(g_1[1]) \to C(g_2[1])$ such that the tangent map $T(f) := f_1$ is a quasi-isomorphism. This definition agrees with the standard definition of a quasi-isomorphism of DGLAs. Sometimes we will call it the *tangent quasi-isomorphism* of the coalgebras corresponding to $g_i, i = 1, 2$.

Suppose that we are given L_∞-algebras g_1 and g_2 and a homomorphism $f : C_1 \to C_2$ of the corresponding dg-coalgebras. We have seen that the morphism of tangent spaces $T(f) : T_0(C_1) \to T_0(C_2)$ is a morphism of complexes. On the other hand $C_i, i = 1, 2$ are complexes and f is a morphism of complexes.

THEOREM 2.7.5. *If $T(f)$ is a quasi-isomorphism then f is a quasi-isomorphism*

Proof. We need to prove that f induces an isomorphism on the cohomology. The complexes $C(g_i[1]), i = 1, 2$ are filtered with the filtrations $F_0^{(i)} \subset F_1^{(i)} \subset ..., i = 1, 2$ where $F_m^i = \oplus_{0 \le j \le m} S^j(g_i[1]), i = 1, 2$. The morphism f is compatible with the filtrations. Let $gr(f)$ be the associated morphism of graded objects (which are complexes as well).

The theorem is a corollary of the following two lemmas.

Lemma 1. Let $h : X \to Y$ be a quasi-isomorphism of complexes. Then its symmetric powers $S^n(h)$ are quasi-isomorphisms.

Lemma 2. If X and Y are filtered complexes with filtrations bounded from below, and $f : X \to Y$ is a morphism preserving filtrations such that $gr(f)$ is a quasi-isomorphism, then f is a quasi-isomorphism.

Let us explain how Lemma 1 and Lemma 2 imply the theorem.

Le us consider the complexes $\oplus_{n \ge 0}(S^n(g_1[1]), Q_1^{(1)})$ and $\oplus_{n \ge 0}(S^n(g_2[1]), Q_1^{(2)})$. Here $Q_1^{(i)}, i = 1, 2$ are the differentials induced by the differentials on the tangent complexes $g_i[1], i = 1, 2$. The latter differentials are first Taylor coefficients of the odd vector fields $Q^{(i)} = Q_1^{(i)} + Q_2^{(i)} + ..., i = 1, 2$. Since $f_1 = T(f)$ is a quasi-isomorphism, we conclude (using Lemma 1) that the induced morphism of the symmteric powers is a quasi-isomorphism as well.

On the other hand, let us consider filtrations of the coalgebras $C(g_i[1]), i = 1, 2$ given by $F_m = \oplus_{0\le n\le m} S^n$ in each case. In fact we have filtrations of the complexes $(C(g_i[1]), Q^{(i)}), i = 1, 2$. The corresponding associated graded complexes are of the type $(S^n(g_i[1]), Q_1^{(i)}), i = 1, 2$. They are quasi-isomorphic by Lemma 1. Applying Lemma 2 we conclude that the morphism f is a quasi-isomorphism.

Proof of Lemma 1

We define a homotopy between morphisms of complexes in the standard way. Namely h is a homotopy between f and g if $[d, h] = f - g$. One writes $f \sim g$ if f is homotopic to g. Two complexes are homotopy equivalent if there exist morphisms f and g such that $fg \sim id$ and $gf \sim id$. One can prove that, for complexes over a field, a quasi-isomorphism is the same as a homotopy equivalence. On the other hand, one can prove that tensor powers of a homotopy equivalence are homotopy equivalences. This proves Lemma 1.

Proof of Lemma 2

Usually such things can be proved by means of spectral sequences, but there is another way outlined below.

Sublemma 1. Morphism $f : X \to Y$ is a quasi-isomorphism iff its cone is acyclic, where the cone is the total complex of the bicomplex

$0 \to X \to Y \to 0 \to ...$, where X is in degree -1.

Sublemma 2. Suppose that X is filtered complex with filtration bounded from below. If $gr(X)$ is acyclic then X is acyclic.

EXERCISE 2.7.6. Prove that the sublemmas imply the Lemma 2.

Proof of Sublemmas

For the first Sublemma, one uses the standard exact sequence:

$$H^i(X) \to H^i(Y) \to H^i(\mathrm{Cone}(f)) \to H^{i+1}(X) \to ...$$

For the second one, one uses the fact that a filtration of the complexes induces the filtration on cohomology. Since the filtration of X is bounded from below we can use induction in order to finish the proof. To deduce Lemma 2 from sublemmas, notice that $gr(Cone(f))$ is acyclic. ■

3. Homotopy classification of formal pointed dg-manifolds and L_∞-algebras

One reason for introducing L_∞-algebras is the following result, which we will prove soon: if there exists t-qis: $C_1 \to C_2$ of the corresponding coalgebras then there exists (non-canonical) t-qis: $C_2 \to C_1$. This is not true in the category of DGLAs. This result imples that t-qis is an equivalence relation. We call it *homotopy equivalence* of L_∞-algebras. Then it is natural to pose the following problem.

Problem: *Classify L_∞-algebras up to homotopy equivalence.*

To solve it we introduce two types of L_∞-algebras.

DEFINITION 3.0.7. An L_∞-algebra is called:

1) linear contractible, if there are coordinates in which $Q_k = 0$ for $k > 1$ and $Ker\, Q_1 = Im\, Q_1$.

2) minimal, if $Q_1 = 0$ in some (equivalently, any) coordinates.

The property to be minimal is invariant under L_∞-isomorphisms, but the property to be linear contractible is not.

DEFINITION 3.0.8. We call a formal pointed dg-manifold contractible if it can be modelled by an L_∞-algebra, which is isomorphic to a linear contractible one. We call a formal pointed dg-manifold minimal if the corresponding L_∞-algebra is minimal.

Now we can state the following important fact.

THEOREM 3.0.9. *Every formal pointed dg-manifold is isomorphic to a direct product of a contractible one and a minimal one.*

DEFINITION 3.0.10. The minimal factor of the product is called the minimal model of the formal pointed dg-manifold (or the corresponding L_∞-algebra).

The above theorem is called the *minimal model theorem.* We are going to prove it in the next subsection. We will finish this subsection by proving the promised result about inversion of tangent quasi-isomorphisms.

COROLLARY 3.0.11. *If $f : g_1 \to g_2$ is a quasi-isomorphism of L_∞-algebras then there is a quasi-isomorphism of L_∞-algebras $h : g_2 \to g_1$.*

Proof. Let g be an L_∞-algebra and g^{min} be a minimal L_∞-algebra from the direct sum decomposition theorem. Then there are two L_∞-morphisms (projection and inclusion)

$$(g[1]_{formal}, 0) \to (g^{min}[1]_{formal}, 0),\ \ (g^{min}[1]_{formal}, 0) \to (g[1]_{formal}, 0)$$

which are quasi-isomorphisms. From this follows that if

$$(g_1[1]_{formal}, 0) \to (g_2[1]_{formal}, 0)$$

is a quasi-isomorphism then there exists a quasi-isomorphism

$$(g_1^{min}[1]_{formal}, 0) \to (g_2^{min}[1]_{formal}, 0)\ .$$

Any quasi-isomorphism between minimal L_∞-algebras is invertible, because it induces an isomorphism of spaces of generators (the inverse mapping theorem).

Then we have an L_∞-isomorphism h which is a composition of this inverse map $(g_2^{min}[1]_{formal}, 0) \to (g_1^{min}[1]_{formal}, 0)$ with the inclusion to $(g_1[1]_{formal}, 0)$. ■

COROLLARY 3.0.12. *Homotopy classes of L_∞-algebras coincide with the L_∞-isomorphism classes of minimal L_∞-algebras.*

Proof. Same as the proof of Corollary 3.34. ■

REMARK 3.0.13. There is an analogy between the minimal model theorem and the theorem from the singularity theory (see, for example, the beginning of Section 11.1 of the book [AGV]) which says that for every germ f of an analytic function f at a critical point, one can find local coordinates $(x^1, \ldots, x^k, y^1, \ldots, y^l)$ such that $f = const + f_2(x) + f_{\geq 3}(y)$ where f_2 is a nondegenerate quadratic form in x and $f_{\geq 3}(y)$ is a germ of a function in y such that its Taylor expansion at $y = 0$ starts at terms of degree at least 3.

3.1. Proof of the minimal model theorem. The idea is to pick coordinates and try to modify them by higher order corrections, finally getting coordinates (x_i, y_i, z_j) such that $Q = \sum_i x_i \partial/\partial y_i + \sum_{j\geq 1} P_j(z)\partial/\partial z_j$, where $P_j(z)$ is a Taylor series in z_i with the lowest term of degree at least 2.

Let $C(V)$ be the coalgebra corresponding to an L_∞-algebra $(V[-1], Q)$. Let us split the complex (V, Q_1) into a direct sum of two complexes: the one with zero differential and the one with trivial cohomology. If the former is trivial, we are done: $Q_1 = 0$ and V is minimal. If not, we can find coordinates (x_i, y_i, z_j) such that $Q_1 = \sum_i x_i \partial/\partial y_i$. The desired splitting is $V = Span\{z_j\} \oplus Span\{x_i, y_i\}$. Then the first summand carries the trivial differential, the second one has the trivial cohomology with respect to Q_1.

We proceed by induction in degree N of the Taylor coefficients of the vector field Q. Assume that $Q = \sum_i x_i\partial/\partial y_i + \sum_{j\geq 1} P_j^N(z)\partial/\partial z_j +$ higher terms. Here $P_j^N(z)$ are polynomials in z_i containg terms of degrees between 2 and N. Let us denote $\sum_i x_i \partial/\partial y_i$ by Q_1.

Next term in the Taylor expansion is

$$\sum_i A_i(x,y,z)\frac{\partial}{\partial x_i} + \sum_i B_i(x,y,z)\frac{\partial}{\partial y_i} + \sum_j C_j(x,y,z)\frac{\partial}{\partial z_j},$$

where A_i, B_i, C_j are homogeneous polynomials of degree $N+1$.

From the equation $[Q,Q] = 0$ we derive the following identities:

(1) $Q_1(A_i) = 0$; (2) $-A_i + Q_1(B_i) = 0$; (3) $Q_1(C_j) =$ some function $F_j(z)$ ($F_j(z)$ arises from commuting of the middle term in the formula for Q with itself).

If we apply a diffeomorphism close to the identity, that is $exp(\xi)$ where ξ is a vector field

$$\xi = \sum_i A_i'\frac{\partial}{\partial x_i} + \sum_i B_i'\frac{\partial}{\partial y_i} + \sum_j C_j'\frac{\partial}{\partial z_j},$$

where A_i', B_i', C_j' are polynomials of degree $N+1$, the change of Q will be:

(a) $A_i \to A_i + Q_1(A_i')$

(b) $B_i \to B_i + A_i' + Q_1(B_i')$

(c) $C_j \to C_j + Q_1(C_j')$

We pose $A_i' := -B_i, B_i' := 0$, thus killing A_i and B_i. Also, we can find C_j' such that the new C_j is a function in z only. The reason is that on $k[x,y,z]$ the cohomology of Q_1 is isomorphic to $k[z]$. More pricisely, LHS in (3) depends on x_i but the RHS does not. Therefore $F_j = 0$, and C_j is cohomologous to an element from $k[z]$. This means that we can find C_j' in (c) such that $C_j + Q_1(C_j')$ depends on z only. Clearly new $C_j(z)$ does not have terms of degree less than 2 in z_j.

Finally we obtain a new coordinate system (x_i, y_i, z_j) such that $Q = \sum_i x_i\partial/\partial y_i + \sum_{j\geq 1} P_j(z)\partial/\partial z_j$ as desired.

Then the formal pointed submanifolds defined by equations $X : \{z_j = 0\}$, $Y : \{x_i = y_i = 0\}$ give a desired product decomposition. It is easy to see that Y defines a minimal L_∞-algebra and X defines a linear contractible L_∞-algebra. This concludes the proof. ■

3.2. Cofibrant dg-manifolds and homotopy equivalence of dg-algebras. Let (A, d_A) be a dg-algebra, that is a commutative unital algebra in the category $Vect_k^{\mathbf{Z}}$.

DEFINITION 3.2.1. We say that A is cofibrant if the following two conditions are satisfied:

a) it is free as a graded commutative algebra, i.e. A is isomorphic to the symmetric algebra $Sym(V), V \in Vect_k^{\mathbf{Z}}$;

b) V is filtered, i.e. $V = \bigcup_{n \geq 0} V_{\leq n}$, where $0 = V_{\leq 0} \subset V_{\leq 1} \subset ...$ is an increasing filtration of graded vector spaces such that $d_A(V_{\leq n}) \subset V_{\leq n-1}$.

Any dg-algebra satisfying the condition a) gives rise to a dg-manifold $(X = Spec(A), Q_X)$. It is given by a (non-formal) smooth scheme X in the tensor category $Vect_k^{\mathbf{Z}}$, as well as a vector field Q_X on X induced by the differential d_A. Because of the condition $d_A(V_{\leq 1}) = 0$ we conclude that Q_X has zero, thus we have a pointed dg-manifold. We will call such pointed dg-manifolds *fibrant*. Fibrant dg-manifolds form a category dual to the category of cofibrant dg-algebras. In order to simplify the notation we will often skip the vector field Q_X, thus writing X instead of (X, Q_X).

DEFINITION 3.2.2. Let $X = Spec(A)$ and $Y = Spec(B)$ be two fibrant dg-manifolds. We say that homomorphisms $f_0 : A \to B, f_1 : A \to B$ (or induced morphisms of fibrant dg-manifolds) are homotopy equivalent if there exists a homomorphism $H : A \to B \otimes \Omega^\bullet(\Delta^1)$ of dg-algebras such that $pr_0 \circ H = f_0, pr_1 \circ H = f_1$. Here $\Omega^\bullet(\Delta^1) \simeq k[t, dt]$ is the algebra of polynomial differential forms on the 1-simplex $[0,1]$, and $pr_i, i = 0,1$ are two projections of this algebra to the field k, such that $pr_i(t) = i, pr_i(dt) = 0$.

PROPOSITION 3.2.3. *Homotopy equivalence is an equivalence relation on homomorphisms of cofibrant algebras.*

Proof. The only non-trivial part is the transivity condition. Assume that f_0 is homotopic to f_1 and f_1 is homotopic to f_2. Let us prove that f_0 is homotopic to f_2. In order to do this we consider a simplicial complex which is the union of two segment $I_1 \cup I_2$ of the boundary of the standard 2-simplex Δ^2. Let $x \in I_1 \cap I_2$ be the only common vertex. The algebra of polynomial differential forms $\Omega^\bullet(I_1 \cup I_2)$ consists of differential forms on I_1 and I_2 which coincide at x. Since $I_1 \cup I_2$ is homotopy equivalent to Δ^2 we have a natural quasi-isomorphism of dg-algebras $\Omega^\bullet(I_1 \cup I_2) \to \Omega^\bullet(\Delta^2)$ induced by the obvious retraction $\Delta^2 \to I_1 \cup I_2$. On the other hand, we have the retraction $\Delta^2 \to I_3$ to the remaining side of Δ^2. In gives a quasi-isomorphism $\Omega^\bullet(I_3) \to \Omega^\bullet(\Delta^2)$. This chain of quasi-isomorphisms induces homotopy equivalences to zero of $f_0 - f_1$ and $f_1 - f_2$. Hence f_0 is homotopy equivalent to f_2. This concludes the proof. ■

REMARK 3.2.4. a) There is an analog of this theorem for non-commutative dg-algebras.

b) One can construct Kan simplicial set $A \otimes \Omega^\bullet(\Delta^n)$, where $\Delta^n, n \geq 0$ is the standard simplex. It follows from the proof that if f_0 is homotopy equivalent to f_1 then there are homomorphisms $H_n : A \to B \times \Omega^\bullet(\Delta^n), n \geq 0$ of dg-algebras such that the composition $pr_i \circ H_n, i = 0,1$ coincide with f_0 and f_1 respectively (here pr_i denotes the projection to the marked vertex $t = i$ of $\Delta^1 \subset \Delta^n$. In other words, homotopy equivalent homomorphisms induce homotopy equivalence of Kan simplicial sets.

c) We can replace ordinary dg-algebras by topologically complete dg-algebras. In this way we get the notion of homotopy equivalent formal dg-manifolds.

3.3. Massey operations. If A is a DGLA then we can construct a structure (unique up to an isomorphism) of a minimal L_∞-algebra on its cohomology $H(A)$ taken with respect to the differential Q_1.

In this case $Q_2 = [\ ]_2$ is the usual bracket on $H^\bullet(A)$. Higher brackets Q_3, Q_4 etc. depend on a choice of coordinates. Only leading coefficients are canonically defined.

The higher brackets can be compared with the so-called Massey products in $H^\bullet(A)$. We give an example of the simplest Massey product of three elements. We take homogeneous $x, y, z \in H^\bullet(A)$ such that $[x, y] = [y, z] = [z, x] = 0$. We want to construct an element in $H^\bullet(A)/\{$ Lie ideal generated by x, y, z $\}$. It will have degree equal to $deg\, x + deg\, y + deg\, z - 1$. Here is a construction. Pick representatives X, Y, Z of x, y, z in $\mathrm{Ker} Q_1$. Then for some α, β, γ we have $[X, Y] = Q_1\gamma, [Y, Z] = Q_1\alpha, [Z, X] = Q_1\beta$. By Jacobi identity: $Q_1([\alpha, X] \pm [\beta, Y] \pm [\gamma, Z]) = 0$. The cohomology class of the expression in brackets is denoted by $[x, y, z]$. This is the triple Massey product.

Exercise 3.3.1. Prove that $[x, y, z]$ is well-defined modulo $[H^\bullet(A), \langle x, y, z\rangle]$ and it is represented by $[x, y, z]_3$ in any coordinate system.

4. Deformation functor

4.1. Groupoid and the foliation in the case of supermanifolds. Geometrically an L_∞-algebra is a formal graded manifold with a marked point, and an odd vector field Q such that $[Q, Q] = 0$ and Q vanishes at the point. In the next subsection we are going to associate with these data a deformation functor. The construction has geometric meaning and can be performed in the case of supermanifolds. In this case all objects can be defined globally, while in the case of formal pointed dg-manifolds we will have to speak about formal graded schemes (in fact small schemes) as functors $Artin_k \to Sets$.

Let S be the subset of zeros of the odd vector field Q on a supermanifold M. Equivalently, $x \in S$ iff $Q(f)(x) = 0$ for all smooth functions f. The (even part of the) space S can be singular. We will ignore this problem for a moment and, assuming that S is a super submanifold of M, we are going to construct a foliation of S. The operator $[Q, \bullet]$ is a differential on the vector fields. Its kernel consists of vector fields commuting with Q. They are tangent to S, and hence define vector fields on S.

We have a sequence $\mathrm{Im}[Q, \bullet] \to \mathrm{Ker}[Q, \bullet] \to \mathrm{Vect}(S)$ of inclusions of real vector spaces. In fact, they are monomorphisms of Lie algebras (by the Jacobi identity), which are $O(S)$-linear (by the Leibniz formula). We are particularly interested in the foliation defined by $\mathrm{Im}[Q, \bullet]$.

We can decompose the even part of S into the union of "leaves", which are subvarieties S_α. Two points belong to the same leaf if they can be connected by smooth curve tangent to a vector field from $Im[Q, \bullet]$.

The formal neighborhood of a smooth point $x \in S$ gives rise to an L_∞-algebra. To be more precise it is a $\mathbf{Z}_2$-graded version of an L_∞-algebra. The formal pointed dg-manifolds corresponding to different points of the same leaf (for the foliation defined by $[Q, \bullet]$) are quasi-isomorphic.

Exercise 4.1.1. Work out the details of the above construction and prove the last statement.

Hint: Use the flows of the vector fields tangent to the leaf.

This geometric picture gives rise to a groupoid in the following way.

1) Objects of the groupoid are points of S.

2) Morphisms between two objects are given by paths $f(t)$ in a leaf and vector fields $v(t)$ such that $f'(t) = [Q, v(t)]$, modulo the following equivalences:

a) $v(t)$ is equivalent to $v(t) + u(t)$ where $u(t)$ vanishes at $f(t)$;

b) $v(t)$ is equivalent to $v(t) + [Q, u(t)]$;

c) the one-parameter group of superdiffeomorphisms $D(t)$ does not change the equivalence class as along as $D(t)$ satisfies the following differential equation

$$d/dt(D(t)x(t)D(t)^{-1}) = [Q, x(t)].$$

EXERCISE 4.1.2. Check that the groupoid axioms are satisfied. (Hint: Look at the super analog of the minimal model for the transverse structure along a leaf. It ensures the local factorization, such that the "trivial" factor is a super analog of the linear contractible formal pointed dg-manifold.)

EXERCISE 4.1.3. The algebra of polyvector fields on a manifold makes the cotangent bundle into a supermanifold (with odd fibers). A Poisson structure is an odd vector field on this manifold. Describe the corresponding groupoid.

4.2. Deformation functor associated with a formal pointed dg-manifold. Now we would like to revisit geometric considerations of the previous section in the case of formal pointed dg-manifolds. Let us sketch what we would like to achieve.

Let C be a cocommutative coalgebra without counit, $Q : C \to C[1]$ be a differential of degree $+1$, R be an Artin algebra with the maximal ideal m.

Points of S (objects of the groupoid) will be $\mathrm{Hom}_{\mathrm{Coalg}}(m^*, C)$ such that the image is contained in the kernel of Q (we take morphisms of graded coalgebras with m placed in degree 0).

In coordinates we have: $C = \mathrm{C}(V[1])$, an object of the groupoid will be $\gamma \in m \otimes V^1$ satisfying the generalized Maurer-Cartan equation:

$$[\gamma]_1 + \frac{1}{2}[\gamma, \gamma]_2 + \frac{1}{6}[\gamma, \gamma, \gamma]_3 + ... = 0.$$

Which objects are equivalent?

Consider the following differential equation for $\gamma(t)$, polynomial in t:

$$\gamma'(t) = [a(t)]_1 + [a(t), \gamma(t)]_2 + \frac{1}{2!}[a(t), \gamma(t), \gamma(t)]_3 + ...,$$

where $a(t)$ is a polynomial in t with values in $V^0 \otimes m$.

We say here that γ_0 is equivalent to γ_1 if there is a solution to this equation such that $\gamma(0) = \gamma_0, \gamma(1) = \gamma_1$.

Morphisms $Hom(\gamma_0, \gamma_1)$ are equivalence classes of such differential equations.

We will spell out this definition similarly to the case of supermanifolds. Namely,

a) We will have an odd vector field Q such that $[Q, Q] = 0$. Zeroes of Q correspond to solutions of the Maurer-Cartan equation.

b) The set of zeros S admits a "foliation" by $[Q, v], v \in Vect(S)$. It gives rise to a holonomy groupoid of the foliation.

c) Moduli space (as a set) is the space of leaves of the foliation.

Let us make all this more precise.

The deformation functor corresponds to a formal pointed dg-manifold M (base point is denoted by 0). The set of solutions to the Maurer-Cartan equation with coefficients in a finite-dimensional nilpotent non-unital algebra m (for example m is the maximal ideal m of R) is defined as the set of m-points of the formal scheme of zeros of Q:

$$Maps\left(\left(Spec(m \oplus k\cdot 1),\text{ base point}\right),\left(Zeroes(Q),0\right)\right) \subset$$

$$Maps\left(\left(Spec(m \oplus k\cdot 1),\text{ base point}\right),\left(M,0\right)\right) \quad .$$

In terms of the coalgebra C corresponding to M this set is equal to the set of homomorphisms of graded coalgebras $m^* \to C$ (m placed in degree 0) with the image annihilated by Q. (Another way to say this is to introduce a *global* pointed dg-manifold of maps from $\left(Spec(m \oplus k\cdot 1),\text{ base point}\right)$ to $(M,0)$ and consider zeros of the global vector field Q on it).

In order to understand the relation to the Maurer-Cartan equation one can observe the following:

a) if $f : m^* \to C(V)$ is a homomorphism of coalgebras and $f_n : m^* \to S^n(V[1])$ its component then:

$f_n = \frac{1}{n!}\sum_{\sigma\in S_n}\sigma\circ f_1^{\otimes n}\circ\Delta^{(n)}$

where $\Delta^{(n)}$ is the iterated coproduct for m^*.

In particular

$$f_n = \frac{f_1\wedge f_1\wedge ...\wedge f_1}{n!}$$

(n wedge factors);

b) the condition $Q(f(x)) = 0$ for any $x \in m^*$ is equivalent to the condition $Q_1(f(x)) + Q_2(f(x)) + ... = 0$ which is the Maurer-Cartan equation

$$[f_1]_1 + \frac{1}{2!}[f_1, f_1]_2 + ... = 0$$

We recall here the well-known lemma.

LEMMA 4.2.1. *(Quillen) if V is a DGLA, then there is a bijection between the set $Hom_{Coalg_k}(m^*, C(V))$ and the set of solutions to the Maurer-Cartan equation $f_1 \in Hom_{Vect_k^{\mathbf{Z}}}(m^*, V[1]) = Hom_{Vect_k}(m^*, V^1) = V^1 \otimes m$.*

Two solutions p_0 and p_1 of the Maurer-Cartan equation are called gauge equivalent iff there exists (parametrized by $Spec(m \oplus k\cdot 1)$) polynomial family of odd vector fields $\xi(t)$ on M (of degree -1 with respect to $\mathbf{Z}$-grading) and a polynomial solution of the equation

$$\frac{dp(t)}{dt} = [Q, \xi(t)]_{|p(t)}, p(0) = p_0, p(1) = p_1,$$

where $p(t)$ is a polynomial family of m-points of formal graded manifold M with base point.

Let us take $(M, 0) = (g[1]_{formal}, 0)$ where g is an L_∞-algebra.

In terms of L_∞-algebras, the set of polynomial paths $\{p(t)\}$ is naturally identified with the set $g^1 \otimes m \otimes k[t]$. Vector fields $\xi(t)$ depend polynomially on t and not necessarily vanish at the base point 0. The set of these vector fields is isomorphic to

$$Hom_{Vect_k^{\mathbf{Z}}}\left(C(g[1]) \oplus (k\cdot 1)^*, g\right) \otimes (m \oplus k\cdot 1)$$

EXERCISE 4.2.2. Check that the gauge equivalence defined above is an equivalence relation. Alternatively, one can define the equivalence relation as the transitive closure of the above relation.

For a formal pointed dg-manifold M we define a set $Def_M(m)$ as the set of gauge equivalence classes of solutions to the Maurer-Cartan equation. The correspondence $m \mapsto Def_M(m)$ extends naturally to a functor denoted also by Def_M. It can be thought of as a functor to groupoids.

DEFINITION 4.2.3. This functor is called deformation functor associated with M.

Analogously, for an L_∞-algebra g we denote by Def_g the deformation functor associated with $(g[1]_{formal}, 0)$.

EXERCISE 4.2.4. Prove the following properties of the deformation functor:

1) For a differential-graded Lie algebra g the deformation functor defined as above for $(g[1]_{formal}, 0)$, is isomorphic to the deformation functor defined in Chapter 1 for DGLAs.

2) Any L_∞-morphism gives rise to a morphism of deformation functors.

3) The functor $Def_{X_1 \times X_2}$ corresponding to the product of two formal pointed dg-manifolds is isomorphic to the product of functors $Def_{X_1} \times Def_{X_2}$,

4) The deformation functor for a linear contractible L_∞-algebra g is trivial, that is $Def_g(m)$ is a one-element set for every m.

Properties 2)-4) are trivial, and 1) is easy. It follows from properties 1)-4) that if an L_∞-morphism of differential graded Lie algebras is a quasi-isomorphism, then it induces an isomorphism of deformation functors.

In the definition of the deformation functor, the finite-dimensional nilpotent commutative algebra m can be replaced by a finite-dimensional nilpotent commutative algebra in $Vect_k^{\mathbf{Z}}$.

LEMMA 4.2.5. *Two maps (inclusion and projection)* $\{$*minimal* $\} \to \{$*minimal* $\} \times \{$*contractible*$\} \to \{$*minimal*$\}$ *induce isomorphisms of deformation functors.*

Proof. This follows from the properties 3) and 4) above. ■

COROLLARY 4.2.6. *Quasi-isomorphisms between L_∞-algebras (resp. DGLA's) induce isomorphisms of the corresponding deformation functors.*

Proof. Using the Lemma we reduce everything to the case of minimal L_∞-algebras. In this case a quasi-isomorphism is the same as an isomorphism. Then the property 2) gives the result. ■

As an illustration of the above Corollary we mention the following theorem of Goldman and Millson.

THEOREM 4.2.7. *The moduli space of representations of the fundamental group of a compact Kähler manifold in a real compact Lie group is locally quadratic.*

This theorem follows from the observation that the DGLA controlling deformations of the representations of the fundamental group is formal, i.e. quasi-isomorphic to its cohomology (as a DGLA). The cohomology has trivial differential, so the Maurer-Cartan equation becomes quadratic: $[\gamma, \gamma] = 0$.

CHAPTER 4

Examples

1. dg-manifolds associated with algebraic examples

In the following subsections we are going to describe DGLAs controlling deformations of associative, Lie and commutative algebras. General technique of the previous chapter allows us to construct the corresponding graded formal pointed dg-manifolds.

1.1. Associative algebras. Let A be an associative algebra without unit. We define the graded vector space of Hochschild cochains on A as

$$C^{\bullet}(A,A) = \oplus_{n\geq 0} Hom_{Vect_k}(A^{\otimes n}, A)$$

and the truncated graded vector space of Hochschild cochains as

$$C^{\bullet}_{+}(A,A) = \oplus_{n\geq 1} Hom_{Vect_k}(A^{\otimes n}, A).$$

Degree of $\varphi \in Hom_{Vect_k}(A^{\otimes n}, A)$ is equal to $deg\, \varphi = n$.

Let $g = g_A = C^{\bullet}(A,A)[1]$ and $g_+ = C^{\bullet}_{+}(A,A)[1]$ be the graded vector spaces with the grading shifted by 1. There is a graded Lie algebra structure on g, so that g_+ is a graded Lie subalgebra. This structure was introduced by Murray Gerstenhaber at the beginning of 60's, so we will call the Lie bracket on g *Gerstenhaber bracket*. It is defined such as follows. First, for any two homogeneous elements φ, ψ such that $deg\, \varphi = n, deg\, \psi = m$ we define their *Gerstenhaber dot product*

$(\varphi \bullet \psi)(a_0, a_1, ..., a_{n+m}) = \sum_{0\leq i\leq n} (-1)^m \varphi(a_0, ..., a_{i-1}, \psi(a_i, a_{i+1}, ..., a_{i+n}), a_{i+n+1}, ..., a_{n+m})$

Then we define the Gerstenhaber bracket

$$[\varphi, \psi] = \varphi \bullet \psi - (-1)^{nm} \psi \bullet \varphi.$$

EXERCISE 1.1.1. Prove that in this way we obtain a graded Lie algebra structure on $g = \oplus_{n\geq -1} g^n$, and $g_+ = \oplus_{n\geq 0} g^n_+$ is a graded Lie subalgebra of g.

So far we did not use an algebra structure on A. We have a multiplication $m : A \otimes A \to A$, hence $m \in g^1_+$. Then we define *Hochschild differential* $d = [m, \bullet]$. Since $[m,m] = 0$ (check this) then one has $d^2 = 0$. The pair (g,d) is called (full) *Hochschild complex*, while the pair (g_+, d) is called (truncated) Hochschild complex. Often this terminology applies directly to $C^{\bullet}(A,A)$ and $C^{\bullet}_{+}(A,A)$.

EXERCISE 1.1.2. Write down Hochschild differential explicitly. Compare with the formulas of Chapter 1, Section 1.1. Check that in this way we obtain a DGLA (g,d) and its sub-DGLA (g_+, d)

There is another, more geometric, version of the these differential-graded Lie algebras. Let us consider the tensor coalgebra $T(A[1]) = \oplus_{n\geq 0}(A[1])^n$. The coproduct is given by $\Delta(1) = 1 \otimes 1, \Delta(v) = v \otimes 1 + 1 \otimes v, v \in A[1]$ and $\Delta(v_1 \otimes ... \otimes v_n) =$

$\sum_{1\le i\le n}(v_1\otimes...\otimes v_i)\bigotimes(v_{i+1}\otimes...\otimes v_n)$, where $v_i\in A[1], 1\le i\le n$. Similarly one defines the truncated tensor coalgebra $T_+(A[1])=\oplus_{n\ge 1}(A[1])^n$. The only difference in formulas is that in the latter case we set $\Delta(v)=0$ for $v\in A[1]$.

Let us denote by $Der(T(A[1])$ and $Der(T_+(A[1])$ the graded Lie algebras of derivations of the above coalgebras. Recall that a derivation D of a coalgebra B has degree n if it is an automorphism of the coalgebra $B\otimes k[\varepsilon]/(\varepsilon^2)$, where $deg\,\varepsilon=n$.

EXERCISE 1.1.3. Consider topological dual tensor algebras $T(A[1])^*$ and $T(A[1])^*_+$. Check that derivations of degree n of the coalgebras correspond to such continuous linear maps of these algebras that $D(ab)=D(a)b+(-1)^{n deg\,a}aD(b)$ (i.e. derivations of degree n of the algebras). In case of $T(A[1])^*$ we also require $D(1)=0$.

The following Proposition is easy to prove, so we leave the proof to the reader.

PROPOSITION 1.1.4. *Graded Lie algebra $Der(T(A[1])$ is isomorphic to g, while $Der(T_+(A[1])$ is isomorphic to g_+. Introducing the Hochschild differential $d=[m,\bullet]$ we obtain DGLAs, which are full and truncated Hochschild complex respectively.*

Having a DGLA g_+ (we skip the differential from the notation), we obtain, as in Chapter 3, the deformation functor $Def_{g_+}: Artin_k\to Sets$. On the other hand we have the "naive" deformation functor $Def^A: Artin_k\to Sets$ such that $Def^A(R)$ consists of associative R-algebras V such that reduction of V modulo the maximal ideal m_R is isomorphic to the k-algebra A. In other words, V is a family of associative algebras over $Spec(R)$ such that the fiber over the point $Spec(k)$ is isomorphic to A. Therefore Def^A describes deformations of the algebra A.

THEOREM 1.1.5. *Functor Def_{g_+} is isomorphic to Def^A.*

Proof. Associative product on a vector space V is an element m of a DGLA $C^\bullet_+(V,V)[1]$ such that $[m,m]=0$. Let now m_0 be an associative product on A. Then, for an Artin algebra R with the maximal ideal m_R we have an element $m\in g^1_A\otimes m_R$ satisfying Maurer-Cartan equation, which in this case says that $[m,m]=0$. Hence we have a family $A\otimes R$ of associative algebras, such that the reduction modulo m_R is isomorphic to A. This gives a morphism of functors $Def_{g_+}\to Def^A$. Conversely, a product m on the algebra V gives a product on A, hence the solution to the Maurer-Cartan equation. In this way we obtain an inverse functor $Def^A\to Def_{g_+}$. ■

This theorem explains why we say that the DGLA g_+ controls deformations of an associative algebra A.

1.2. Lie algebras. Deformation theory of Lie algebras is similar to the deformation theory of associative algebras. For a Lie algebra W over k we consider the graded vector space of cochains $C^\bullet(W,W)=\oplus_{n\ge 1}Hom(\bigwedge^n W,W)$. The shifted graded vector space $g=g_W=C^\bullet(W,W)[1]$ has the natural graded Lie algebra structure, if we interpret it as a graded Lie algebra of graded derivations of the cocommutative coalgebra $S(W[1])=\oplus_{n\ge 1}S^n(W[1])$ (see Chapter 3). Similarly to the case of associative algebras the Lie bracket $b:\bigwedge^2 W\to W$ gives rise to a DGLA structure on g. Corresponding complex is called *Chevalley complex* of the Lie algebra W. Therefore we have a deformation functor $Def_g: Artin_k\to Sets$. On the other hand we have a "naive" deformation functor $Def^W: Artin_k\to Sets$ which assigns to an Artin algebra R a family α of Lie algebras over R, such that the reduction modulo the maximal ideal m_R is isomorphic to W. We leave to the reader to prove the following result.

THEOREM 1.2.1. *Functors Def_g and Def^W are isomorphic.*

We say that the DGLA $g = g_W$ controls the deformation theory of W.

1.3. Commutative algebras. Let A be a commutative algebra over k. Construction of the DGLA controlling deformations of A is more complicated in this case. First, we observe that for any vector space V there is a cocommutative Hopf algebra structure on the tensor coalgebra $T(V)$. The coproduct is uniquely determined by the formulas $\Delta(1) = 1 \otimes 1, \Delta(x) = x \otimes 1 + 1 \otimes x, x \in V$. We skip formulas for the product (it is called shuffle-product). In any case we have a cocommutative Hopf algebra $T(V)$. Therefore it is isomorphic to the universal enveloping algebra $U(L(V))$ of some Lie algebra $L(V)$.

EXERCISE 1.3.1. Show that $L(V) = V \oplus \bigwedge^2 V \oplus V^{\otimes 3}/J \oplus ...$, where $J = Coker(\bigwedge^3 V \to V \otimes \bigwedge^2 V)$ is the cokernel of the linear map $a \wedge b \wedge c \mapsto a \otimes (b \wedge c) + b \otimes (a \wedge c) + c \otimes (a \wedge b)$.

We define the graded Lie algebra g_V of graded derivations of the cocommutative Hopf algebra $T(V)$. Finally, we apply this construction to $V = A[1]$. Then $g = g_A = \oplus_{n \geq 0} g^n$, where $g^0 = Hom_{Vect_k}(A, A), g^1 = Hom_{Vect_k}(S^2(A), A), g^3 = Hom_{Vect_k}(Coker(S^3(A) \to A \otimes S^2(A)), A)$, etc. In fact this graded Lie algebra is a graded Lie subalgebra of Hochschild cochains of A, considered as an associative algebra. We are going to denote g by g^{Har}, so it will not be confused with the graded Lie algebra of Hochschild cochains.

PROPOSITION 1.3.2. *The condition $[\gamma, \gamma] = 0$ for $\gamma \in g^1$ is equivalent to the fact that γ is a commutative associative product on A.*

Proof. The image of the natural embedding of g^{Har} into the graded Lie algebra of Hochschild cochains of A consists of derivations which preserves not only coproduct, but also a shuffle-product. The results follows. ■

Therefore, having a commutative associative product on A we can make $g^{Har} = g_A^{Har}$ into a DGLA. The corresponding complex is called *Harrison complex*.

Similarly to the case of associative algebras we can define two functors: $Def_{g^{Har}} : Artin_k \to Sets$ and $Def^A : Artin_k \to Sets$, which is the "naive" deformation functor (we leave the definition to the reader). Again we have the following theorem.

THEOREM 1.3.3. *Functors $Def_{g^{Har}}$ and Def^A are isomorphic.*

We say that DGLA g_A^{Har} controls deformations of A as a commutative associative algebra.

REMARK 1.3.4. The reader can easily see that all the deformation functors in this section are in fact functors to groupoids. All three theorems give equivalence of functors from Artin algebras to groupoids.

2. dg-manifolds associated with geometric examples

We are going to discuss geometric examples. Here we have a new phenomenon: formal pointed dg-manifolds can be derived directly from the problem, without introducing a DGLA first.

2.1. Systems of polynomial equations. Let M be a smooth manifold, $f_1, ..., f_n : M \to \mathbf{R}$ a finite collection of smooth functions. Let us fix $m_0 \in M$ and consider the following problem: how to deform solutions to the system of equations

$$f_1(m) = f_2(m) = ... = f_n(m) = 0?$$

In particular, we can ask about the corresponding functor $Artin_{\mathbf{R}} \to Sets$.

Let us consider the formal completion $\widehat{M}_{m_0}$ of M at m_0. Then we can lift given functions to the formal functions $\widehat{f}_i, 1 \le i \le n$.

We have a functor F which assigns to an Artin algebra R the set of all morphisms $\pi : Spec(R) \to \widehat{M}_{m_0}$ such that $\pi^*(\widehat{f}_i) = 0, 1 \le i \le n$.

REMARK 2.1.1. More generally, we can have a formal manifold over any field and any number of regular functions on it.

By the Schlessinger theorem (see Chapter 1) functor F is represented by an ind-scheme over $\mathbf{R}$. This ind-scheme can be singular. The idea is to find an L_∞-algebra g such that Def_g is isomorphic to F, but the formal moduli space associated with g is non-singular.

Namely, let us define $\widehat{M}^{odd}_{m_0} = \widehat{M}_{m_0} \times Spec(\mathbf{R}[\xi_1, ..., \xi_n])$ such that $deg\, \xi_i = -1, 1 \le i \le n$. Then $\widehat{M}^{odd}_{m_0}$ is a formal graded manifold.

EXERCISE 2.1.2. Prove that $d = \sum_{1 \le i \le n} f_i \partial / \partial \xi_i$ is a vector field on $\widehat{M}^{odd}_{m_0}$ of degree $+1$, such that $[d, d] = 0$.

Let us consider a formal scheme Z of common zeros of $\widehat{f}_i, 1 \le i \le n$. We can think of Z as a functor $Artin_{\mathbf{R}} \to Sets$. This is the functor F we mentioned above.

This picture can be generalized further. Namely, let $\widehat{M}$ be a formal manifold over a field k, and $E \to \widehat{M}$ be a vector bundle (it is given by a finitely generated projective $\mathcal{O}(\widehat{M})$-module). Let us fix a section $s \in \Gamma(\widehat{M}, E)$. Then one has an ind-scheme $Z(s)$ of zeros of s. As a functor $Artin_k \to Sets$ it can be described such as follows. Let us consider the total space of the formal $\mathbf{Z}$-graded manifold $tot(E[-1])$. This formal graded manifold carries a vector field d_{tot} of degree $+1$ such that $[d_{tot}, d_{tot}] = 0$. If one trivializes the vector bundle, then, in local coordinates $d_{tot} = \sum_i f_i \partial / \partial \xi_i$, where $s = (f_1, ..., f_n)$ and $\xi_i, 1 \le i \le N$ are coordinates along the fiber of $E[-1]$.

DGLA (more precisely, the complex) corresponding to the formal pointed dg-manifold $Z(s)$ is called the Koszul complex. If E is a trivial vector bundle, we obtain the previous case of a collection of functions. The following proposition is a direct reformulation of definitions.

PROPOSITION 2.1.3. *One has an isomorphism of vector spaces* $H^0(\mathcal{O}(tot(E[-1])), d_{tot}) \simeq \mathcal{O}(Z(s)) \simeq \mathcal{O}(\widehat{M})/(f_1, ..., f_n)$.

In fact this proposition holds in infinite-dimensional case as well.

One can ask whether higher cohomology groups vanish. We will present without a proof the result in the finite-dimensional case.

THEOREM 2.1.4. *The following conditions are equivalent:*
a) $dim(Z(s)) = dim(\widehat{M}) - rk(E)$,
b) $H^i(\mathcal{O}(tot(E[-1])), d_{tot}) = 0$ *for* $i < 0$,

c) there exists a trivialization of E such that $s = (f_1, ..., f_n)$ and f_i is not a zero divisor in $\mathcal{O}(\widehat{M})/(f_1, ..., f_{i-1})$ for all $1 \leq i \leq n$ (we assume $f_0 = 0$).

If the condition c) is satisfied, the section s (or the corresponding sequence of functions) is called *regular* (or complete intersection).

REMARK 2.1.5. Suppose $E \to \widehat{M}$ is a vector bundle over an ind-affine scheme. We say that an ind-subscheme $Z \subset M$ is an abstract complete intersection if $\mathcal{O}(Z) \simeq H^0(\mathcal{O}(tot(E[-1])), d_{tot})$. One can prove that realization of Z as an ind-scheme of zeros of a section of E described above is unique up to a quasi-isomorphism of the corresponding formal pointed dg-manifolds.

2.2. Group action. Instead of the scheme of zeros $f_i(m) = 0, 1 \leq i \leq n$ we can consider a quotient under the Lie group action. Namely, let G be a finite-dimensional Lie group, $g = Lie(G)$ be its Lie algebra. Suppose that G acts on a formal manifold X. Let us consider a (partially) formal $\mathbf{Z}$-graded manifold $g[1] \times X$. It is understood as a functor from $\mathbf{Z}$-graded Artin algebras to $Sets$. There is a vector field d of degree $+1$ on $g[1] \times X$. Namely, $d(\gamma, x) = (\frac{1}{2}[\gamma, \gamma], v_\gamma(x))$, where $v_\gamma(x)$ is the vector field on X generated by $\gamma \in g$.

EXERCISE 2.2.1. Check that $[d, d] = 0$.

Therefore we have a formal dg-manifold and can consider the formal scheme of zeros $Z(d)$ (it is thought of as a functor $Artin_k \to Sets$). This functor has another description. Namely, let us consider a functor $F : Artin_k \to Sets$ such that $F(R) = X(R)/G(R)$ (i.e. the quotient set of R-points).

Proofs of the following two results are left as exercises to the reader.

PROPOSITION 2.2.2. *Functor F is isomorphic to the functor $Z(d)$.*

PROPOSITION 2.2.3. *If action of G is free then:*

a) the quotient X/G is a formal manifold. It is quasi-isomorphic to the formal pointed dg-manifold $Z(d)$.

b) We have an isomorphism $H^0(\mathcal{O}(g[1] \times X)) \simeq \mathcal{O}^G(X)$ (in the RHS we take the space of invariant functions).

QUESTION 2.2.4. Is it reasonable to consider such actions that $H^{>0}(\mathcal{O}(g[1] \times X) = 0$? It would be an analog of a complete intersection.

2.3. Homotopical actions of L_∞-algebras. Let now g be an L_∞-algebra and X be a formal graded manifold. Then the product $g[1] \times X$ is a formal graded manifold. Let $d_{g[1]}$ denotes the vector field of degree $+1$ induced by the L_∞-structure.

DEFINITION 2.3.1. Homotopical action of g on X is a vector field d of degree $+1$ on $g[1] \times X$ such that $[d, d] = 0$ and the natural projection $g[1] \times X \to g[1]$ is a morphism of formal dg-manifolds.

More generally, let g be an L_∞-algebra and (X, d_X) be a formal dg-manifold.

DEFINITION 2.3.2. A homotopical action of g on X is given by an epimorphism of formal dg-manifolds $\pi : (Z, d_Z) \to (g[1], d_{g[1]})$ together with an isomorphism of formal dg-manifolds $(\pi^{-1}(0), d_Z) \simeq (X, d_X)$.

We can say that g "acts" on the fibers of a formal bundle and the "quotient space" is the total space of the bundle. This situation is similar to the one in topology. If a Lie group G acts on a topological space X, one can define a homotopy quotient of this action as the total space of the bundle $EG \times_G X \to BG$. Notice that if G is compact then $H^\bullet(BG, \mathbf{R}) \simeq H^\bullet(g, \mathbf{R})$. The RHS of this formula is the cohomology of the Chevalley complex $(C^\bullet(g[1]), d)$, hence describes the formal dg-manifold $(g[1], d_{g[1]})$.

REMARK 2.3.3. One can impose at the same time conditions $f_i = 0, 1 \le i \le n$ and factorize by a (homotopical) action of a Lie algebra. In this way one can combine in a single formal dg-manifold all geometric examples discussed above.

2.4. Formal differential geometry. Let $\pi : E \to X$ be a submersion of smooth finite-dimensional manifolds (we will call it a bundle for short). Then we have an infinite-dimensional bundle of infinite jets of sections: $\pi_\infty : J_\infty(E) \to X$. It is well-known that the bundle of infinite jets carries a flat connection ∇_∞, so that flat sections of π_∞ are exactly sections of π. Let us fix $s \in \Gamma(X, E)$ and consider the following problem: *what is the formal pointed dg-manifold controlling the deformation theory of s?*

REMARK 2.4.1. More generally, we can assume that $\pi : E \to X$ is a submersion (bundle) of supermanifolds (it is the same as a submersion of the underlying even manifolds).

For each $k \ge 0$ we have a bundle of k-jets $\pi_k : J_k(E) \to X$. For any open $U \subset J_k(E)$ we have an infinite-dimensional manifold $\Gamma_{(U)}(X, E)$. It consists of $s \in \Gamma(X, E)$ such that the k-jet of s belongs to U.

EXERCISE 2.4.2. Describe this manifold as a functor on real Artin algebras.

Let $T_{E|X}$ be a vertical tangent bundle, and $v \in \Gamma(U, \pi_k^*(T_{E|X}))$ a vector field. Then we can construct a canonical vector field v_s on $\Gamma_{(U)}(X, E)$. Indeed, let $s \in \Gamma_{(U)}(X, E)$. We can consider the bundle which is the pull-back $(j_k(s))^* \pi_k^*(T_{E|X}) = s^*(T_{E|X}) \to X$.

PROPOSITION 2.4.3. *One has the following isomorphism*

$$T_s(\Gamma_{(U)}(X, E)) \simeq \Gamma(X, s^*(T_{E|X})),$$

where T_s denotes the tangent bundle at s.

Proof. Clear. ■

Therefore, we can define $v_s = (j_k(s))^*(v)$.

DEFINITION 2.4.4. Such vector fields are called local vector fields (or vector fields given by local differential expressions).

It often happens in practice that v_s has degree $+1$ and $[v_s, v_s] = 0$. Then we have a formal pointed dg-manifold $\mathcal{M}_s$, which is a formal neighborhood of s in $\Gamma_{(U)}(X, E)$.

REMARK 2.4.5. It often happens (and it is a wish in general) that the tangent complex $T_s(\mathcal{M}_s)$ is elliptic. This explains why often overdetermined systems of differential equations have a solution. Among examples are complex structures, metrics with special holonomy, etc.

Replacing a manifold X by the union $\widehat{X}$ of formal neighborhoods of its points, we get a formal bundle over $\widehat{X}$. Then we can repeat the above construction and obtain a vertical vector field $\widehat{v}_s$ on the bundle of infinite jets π_∞. It is easy to see that $\widehat{v}_s(j_\infty(s)) = 0$. We will assume that $deg(\widehat{v}_s) = +1$ and $[\widehat{v}_s, \widehat{v}_s] = 0$. One can see that $\widehat{v}_s$ is covariantly constant with respect to the connection ∇_∞. If we recall that $j_\infty(s)$ is a flat section of π_∞, we see that the deformation problem is reformulated in terms of formal differential geometry. In this way we obtain the formal pointed dg-manifold $\mathcal{M}_s^\infty$ controlling the deformation theory of s.

There is a quasi-isomorphic formal pointed dg-manifold, which is sometimes easier to use. In order to describe it we recall that the total space $T[1]X$ of the tangent odd bundle carries an odd vector field d_{dR} of degree $+1$ such that $[d_{dR}, d_{dR}] = 0$. It comes from the de Rham differential on differential forms. Let us consider the pull-back of the bundle π_∞ to $T[1]X$. The total space of the pull-back carries an odd vector field ξ of degree $+1$ such that $[\xi, \xi] = 0$. It is a sum of d_{dR} and the pullback of $\widehat{v}_s$. We have also a pull-back of $j_\infty(s)$. We can deform this section as a section of formal supermanifolds. It is a zero of ξ. Thus we have a formal pointed dg-manifold $\mathcal{N}_s^\infty$.

Proposition 2.4.6. *There is natural quasi-isomorphism of formal pointed dg-manifolds* $\mathcal{M}_s^\infty \to \mathcal{N}_s^\infty$.

Proof. To a section of $E \to X$ we associate its pull-back to $T[1]X$. In order to finish the proof we observe that $\Omega^*(X, T_{j_\infty(s)}(\Gamma(X, J_\infty(E)))$ is quasi-isomorphic to $\Gamma(X, E)$. ■

Remark 2.4.7. We can construct formal pointed dg-manifolds in the cases when no bundle is apparent. For example we can ask about deformations of an embedding of smooth manifolds $j : M \to N$. This problem reduces to the deformation theory of the induced section of the bundle $M \times N \to M$.

We will discuss more examples in the next setion.

3. BRST

3.1. General scheme. Let $E \to X$ be a non-linear bundle with the fibers which are supermanifolds, and with the base X which is an even finite-dimensional manifold. Then for an open $U \subset X$ the space of sections $E(U)$ of E over U is an infinite-dimensional supermanifold. We would like to make it into a dg-manifold. In the considerations below we take $U = X$. The corresponding odd vector field should be given by "local" formulas. In other words, if $s \in E(X) = \Gamma(X, E)$ then $d_{E(X)}(s) \in T_s(E(X)) = \Gamma(X, s^*T_X)[1]$ should depend on a finite jet of the section s. The space of odd vector fields which satisfy the locality condition forms a Lie superalgebra $Vect_{loc}(X)[1] \subset Vect(E(X))[1]$ (in fact we have a sheaf of subalgebras, since our considerations are local). Choosing $d \in Vect_{loc}(X)[1]$ such that $[d, d] = 0$ we obtain a sheaf (on X) of infinite-dimensional dg-manifolds. They are non-formal dg-manifolds. Formal neighborhood of a zero d gives rise to a deformation functor. This is a classical BRST construction.

Example 3.1.1. Let us consider the standard fiber bundle $\pi : E = T[1]X \to X$. Then $\Gamma(X, E) = Vect(X)[1]$ is a dg-manifold. The corresponding odd vector field d is defined by the Lie algebra structure on $Vect(X)$. Let (x_i, ξ^i) be local coordinates on $T[1]X$. A section $s \in \Gamma(X, E)$ is given by $\xi^i = v^i(x)$. Then the odd vector field d is given by the formulas $\dot{v}^i = \sum_i v^j(x)\partial v^i(x)/\partial x^j = 1/2[v,v]^i(x)$. Formal neighborhoods of zeros of d (i.e. points x where $v^i(x) = 0$) define the deformation theory of X as a smooth manifold.

3.2. Jet bundles and non-linear equations.

3.2.1. *Determined systems.* Let $p : Y \to X$ be a (non-linear) fiber bundle of smooth manifolds, and $\pi : V \to Y$ be a vector bundle. We denote by $Jet^N(Y)$ the space of N-jets of smooth sections of $p : Y \to X$. For a given section $s \in \Gamma(X,Y)$ we denote by $j_N(s)$ the corresponding N-jet. Then we have a pull-back vector bundle $\pi^*V \to J_N(Y)$. Let us also fix a section Φ of the pull-back bundle.

Definition 3.2.1. An differential equation of order less or equal than N (for a section $s \in \Gamma(X,Y)$) is given by $\Phi(j_N(s)) = 0$.

Let us introduce a supermanifold $E = V[-1]$. It gives rise to a fiber bundle on X: $E \to Y \to X$. Let $s \in \Gamma(X,E)$. We define $\hat{s} \in \Gamma(X,Y)$ by composing $s : X \to Y$ with the projection $E \to Y$. Then $\Phi(\hat{s}) \in \Gamma(X, \hat{s}^*V) \subset T_s(\Gamma(X,E))$, where $T_s(\Gamma(X,E))$ denotes the tangent space to s in the space of sections. In this way we obtain an odd vector field d such that $[d,d] = 0$. Thus a closed subset in the space of N-jets is described as a set of zeros of a section of some vector bundle on $J_N(Y)$.

Let us explain this point in detail. Suppose that we are given a differential expression $D : J_N(Y) \to \mathbf{R}$. Naively, a non-linear differential equation of order less or equal than N is given by $D(j_N(s)) = 0$. We can say the same thing differently. Let us consider a dg-manifold $\Gamma(X,Y) \times C^\infty(X)[1] = \Gamma(X, Y \times \mathbf{R}^{0|1})$. Local coordinates on it will be denoted by $(s,\xi) = (s(x), \xi(x))$. Then we define an odd vector field d by the formulas $\dot{s}(x) = 0, \dot{\xi}(x) = D(jet^N(s)(x))$. One can check that $[d,d] = 0$. Taking $V = Y \times \mathbf{R}$ we arrive to the previous description. Formal neighborhoods of zeros of d control the deformation theory of *determined* systems of non-linear equations.

3.2.2. *Overdetermined systems.* For overdetermined systems of non-linear equations there are solvability conditions. They give rise to odd variables. Roughly speaking, in this case one works with such DGLAs $g = \oplus_{n \geq 0} g^n$ that g^0 consists of functions (sections of bundles, etc.) , g^1 corresponds to equations, g^2 corresponds to compatibility conditions. Symmetries of the equations appear in g^{-1}.

Let us consider a typical example: integrability conditions for almost complex structures. Thus we have a smooth manifold X, $dim X = 2n$. We also have a bundle $Y \to X$ of almost complex structures. The fiber consists of linear maps $J_x : T_xX \to T_xX$ such that $J^2 = -id$. Equivalently, it is given by the space of vector subspaces $T_x^{1,0} \subset T_xX \otimes \mathbf{C}$ such that $T_x^{1,0} \cap \overline{T_x^{1,0}} = 0$ and $T_x^{1,0} \oplus \overline{T_x^{1,0}} = T_xX \otimes \mathbf{C}$. For any global almost complex structure one can define its curvature as an element of $\Gamma(X, Hom(\bigwedge^2 T^{0,1}, T_X \otimes \mathbf{C}/T_X^{0,1}))$. One can do this adding also forms of higher degrees. Then one gets an elliptic complex. It carries a structure of a DGLA, which corresponds to the (extended) formal moduli space of complex structures. In degree -1 one has symmetries of the integrability conditions. They are *all* smooth vector fileds on X, acting on the space of complex structures.

Remark 3.2.2. One can also have examples in which there is no “honest” Lie algebra of symmetries of equations. In fact one can replace a Lie algebra of symmetries by its resolution and consider homotopy actions. Such example appear indeed when one considers the action of holomorphic vector fields on holomorphic foliations.

3.3. Volume elements. Let us denote by $\Omega^{n,+}(X)$ the cone of positive volume elements on a real smooth n-dimensional manifold X. We put $E = \Omega^{n,+}(X) \times_X T[1]X$. Then $\Gamma(X, E)$ becomes a dg-manifold with an odd vector field $d = (Lie_\xi \omega, [\xi, \xi])$, where $\omega \in \Omega^{n,+}(X), \xi \in T[1]X$. Let us take a section $s(x) = (\omega(x), 0)$. Then the corresponding dg-manifold is related to the Lie algebra of vector fields with zero divergence.

3.4. dg-manifolds associated with deformations of local systems, complex vector bundles and complex manifolds.

3.4.1. *Local systems.* Let X be a smooth manifold, G a Lie group. Let us consider a G-local system on X, which is a smooth principal G-bundle $E \to X$ equipped with a flat connection ∇. Let $g = Lie(G)$ be the Lie algebra of G. Then we have the associated vector bundle $ad(E) \to X$ which carries the induced flat connection. We are interested in the deformation theory of this flat connection.

Let $\Gamma^\bullet = \oplus_{n \geq 0} \Gamma^n$ be a graded Lie algebra $\Gamma(X, ad(E) \otimes \Omega^\bullet(X))$ of differential forms with values in $ad(E)$. The bracket is given locally as on the product of Lie algebra and graded commutative algebra. The flat connection defines a differential d_∇ on Γ.

PROPOSITION 3.4.1. *DGLA* $(\Gamma^\bullet, d_\nabla)$ *controls the deformation theory of* ∇.

Proof. Let R be an Artin algebra, $\nabla + \alpha$ be a flat connection. Then, in terms of $\Gamma^\bullet$, we can say that we have $\alpha \in \Gamma^1 \otimes m_R$ satisfying the Maurer-Cartan equation

$$d_\nabla(\alpha) + 1/2[\alpha, \alpha] = 0.$$

Gauge transformations are given by elements of the group $exp(\Gamma^0 \otimes m_R)$. Therefore the "naive" deformation functor Def^∇ (we leave as an exercise to define it) is isomorphic to the functor $Def_{\Gamma^\bullet}$. ■

Thus we have a formal pointed dg-manifold, which is associated with $\Gamma^\bullet$ and controls the deformation theory of ∇.

The latter result can be reinterpreted in terms of BRST construction. Let us consider the pull-back $F \to T[1]X$ of the supervector bundle $ad(E[1]) \to X$. There is a flat superconnection on F. The total space $tot(F)$ carries a vector field d_{tot} of degree $+1$ such that $d_{tot} = d_{dR} + d_{g[1]}$ in the previous notation. Clearly $[d_{tot}, d_{tot}] = 0$ and ∇ defines a section s of F such that $d_{tot}(s) = 0$. Then, as we know, we have a formal pointed dg-manifold, which is a formal neighborhood of s in the space of sections.

3.4.2. *Complex structures and complex vector bundles.* To a real smooth manifold X we can associate two closely related algebras: the algebra $\mathcal{O}(X)$ of *real valued* smooth functions on X and the algebra $\mathcal{O}(X) \otimes \mathbf{C} := \mathcal{O}(X_{\mathbf{C}})$ of *complex-valued* smooth functions on X. We would like to think of the latter as of the algebra of regular functions on the "very thin" complex extension $X_{\mathbf{C}}$ of X. If X was a real-analytic, then $X_{\mathbf{C}}$ would be a germ of the corresponding complex manifold. In other words, we think of X as of a pair $(X_{\mathbf{C}}, *)$, where $*$ is the complex conjugation.

If X admits a complex structure then the formal completion $\widehat{X_{\mathbf{C}x}}$ at each point x is a product of two formal manifolds: holomorphic and antiholomorphic: $\widehat{X_{\mathbf{C}x}} \simeq \widehat{X_{\mathbf{C}x}}^{hol} \times \widehat{X_{\mathbf{C}x}}^{antihol}$.

Thus we have two complex conjugate formal foliations of $\widehat{X_{\mathbf{C}}}$. Holomorphic vector bundle on X give rise to a holomorphic vector bundle on $\widehat{X_{\mathbf{C}}}$ (i.e. projective finitely-generated $\mathcal{O}(X_{\mathbf{C}})$-module) with a connection, which is flat in the anti-holomorphic direction.

In order to describe the deformation theory of such a connection ∇ one should replace in the previous subsection de Rham differential forms by Dolbeault differential forms.

EXERCISE 3.4.2. a) Prove that the DGLA of Dolbeault forms $g^{\bullet} = \oplus_{n\geq 0}\Gamma(X, \bigwedge^n(T_X^{0,1})^*\otimes T_X^{1,0})$ (differential is induced by the $\bar{\partial}$-operator) controls the deformation theory of the complex structure on X.

b) Let $E \to X$ be a holomorphic vector bundle with holomorphic connection. Let us think of it as of a smooth vector bundle with a connection which is flat in $\bar{\partial}$-direction.

Prove that the DGLA of Dolbeault E-valued forms $g^{\bullet} = \oplus_{n\geq 0}\Gamma(X, E\otimes\Omega^{0,*}(X))$ (differential is induced by the connection) controls the deformation theory of the complex connection on E.

Returning to the BRST picture, we remark that to a smooth manifold X one can associate a supervector bundle over $X_{\mathbf{C}}$ with the total space $Spec(\Omega^{0,*}(X))$ and the fiber over a point x given by $Spec(\bigwedge^{\bullet}(T_x^{0,1})^*)$. Then we can repeat considerations of the previous subsection replacing $T[1]X$ by $T^{0,1}[1]X$. This gives us the formal pointed dg-manifold controlling the deformation theory of the vector bundle with a holomorphic connection.

Finally we observe that, similarly to complex structures, one can treat real foliations. More precisely, let X be a smooth manifold which carries a foliation F. Then F is defined by an integrable subbundle of the tangent bundle T_X. In particular we can consider a supermanifold $T[1]F \subset T[1]X$.

EXERCISE 3.4.3. Check that the odd vector field d_{dR} on $T[1]X$ is tangent to $T[1]F$ (Hint: foliation is defined by a differential ideal J_F in $\Omega^{\bullet}(X)$ which consists of forms vanishing on T_F).

Let $g = Vect(T[1]F)$ be a graded Lie algebra of vector fields on a supermanifold $T[1]F$. It follows from the exercise that $d = [d_{dR}, \bullet]$ makes g into a DGLA. Then we have a deformation functor $Def_g : Artin_{\mathbf{R}} \to Sets$. On the other hand we have a "naive" deformation functor $Def^F : Artin_{\mathbf{R}} \to Sets$ which assigns to an Artin algebra (R, m_R) a class of isomorphism of families of foliations on X parametrized by $Spec(R)$ modulo the gauge action of the group $exp(m_R \otimes Vect(X))$.

EXERCISE 3.4.4. Prove that Def_g is isomorphic to Def^F (Hint: compare with the deformation theory of complex structures, i.e. deform the de Rham differential along the leaves of F).

3.5. Deformations of holomorphic maps. Let $\phi : X \to Y$ be a holomorphic map of complex manifolds. We would like to describe the deformation theory of ϕ. As always, we have a "naive" deformation functor $Def^{\phi} : Artin_{\mathbf{C}} \to Sets$. Namely, $Def^{\phi}(R)$ consists of morphisms of analytic spaces $X \times Spec(R) \to Y$ such that their restriction to $X \times Spec(\mathbf{C})$ coincides with ϕ. We would like to describe a formal pointed dg-manifold M such that the deformation functor Def_M is isomorphic to Def^{ϕ}.

We recall (see Section 3.4) that with a C^∞-manifold Z we can associate its complexification $Z_{\mathbf{C}}$, which is the same real manifold, but equipped with the sheaf of complexified smooth functions $C_Z^\infty \otimes \mathbf{C}$. We will use the same notation in the case when Z is a supermanifold (see Chapter 2, Section 8).

Let us consider a complexified supermanifold $X_1 = (T^{0,1}[1]X)_{\mathbf{C}}$. More precisely, we start with X considered as a smooth real manifold. Then we can define a supermanifold $T^{0,1}[1]X$ which is the total space of the vector bundle of anti-holomorphic vectors, with the changed parity of fibers. Finally we complexify the algebra of functions on this supermanifold. Clearly $C^\infty(X_1) \simeq \Omega^{0,*}(X)$. There is natural action of the complex supergroup $\mathbf{C}^* \times \mathbf{C}^{0|1}$ on X_1. Similarly to the case of ordinary differential forms (see Chapter 2, Section 6.1 and Section 8) now we recover the Dolbeault differential.

Let $E_Y \to X$ be a bundle over X with the fiber $E_{Y,x}$ being the complexification of the formal manifold $(\widehat{Y}_{\phi(x)})_{\mathbf{C}}$, which is the complexification of the completion of C^∞-manifold Y at $\phi(x)$. In fact E_Y carries a flat connection, hence the Lie algebra $Vect(X)$ acts on the space of sections $\Gamma(X, E_Y)$. If Y carries a complex structure, then, as we discussed in Section 3.4 we have a factorization $(\widehat{Y}_{\phi(x)})_{\mathbf{C}} = \widehat{Y}^{hol}_{\phi(x)} \times \widehat{Y}^{antihol}_{\phi(x)}$. We denote by $E_1 \to X_1$ the pull-back to X_1 via ϕ of the bundle $E_Y^{hol} \to X$, obtained from E_Y by taking the factor $\widehat{Y}^{hol}_{\phi(x)}$ for each $x \in X$ in the above factorization. Notice that E_Y^{hol} carries a flat connection (holomorphic functions for infinitesimally closed points can be identified). Finally, we denote by M the formal pointed dg-manifold which is the completion at ϕ of the space of section $\Gamma(X_1, E_1)$ (vector field $d = d_M$ of degree $+1$ such that $[d, d] = 0$ and $d(\phi) = 0$ arises from the action of $\mathbf{C}^* \times \mathbf{C^{0|1}}$, as we discussed above).

THEOREM 3.5.1. *There is an isomorphism of deformation functors $Def_M \simeq Def^\phi$.*

Proof. Let (R, m_R) be a complex Artin algebra. Then $Def_M(R)$ consists of such sections of the bundle $E_2 \to X_1 \times Spec(R)$ (which is the pull-back of E_1 via the natural projection $X_1 \times Spec(R) \to X_1$) that their restriction to $X_1 \times Spec(\mathbf{C})$ coincides with ϕ. Notice that E_2 carries a flat connection. We are interested in $\mathbf{C}^* \times \mathbf{C}^{0|1}$-invariant sections of E_2. It is easy to see that the space $\mathbf{C}^*$-invariant sections of E_2 is isomorphic to the space of smooth sections of the bundle $E_Y^{hol} \to X_{\mathbf{C}} \times Spec(R)$ (recall that $X_{\mathbf{C}}$ denotes the complexification of X considered as a C^∞-manifold). The latter space can be described explicitly. Namely, for any $x \in X$ we choose a small Stein neighborhood U_x as well as a small Stein neighborhood $U_{\phi(x)}$ of $\phi(x)$ such that $\phi(U_x) \subset U_{\phi(x)}$. Then a section $s \in \Gamma(U_x, E_Y^{hol})$ is the same as such homomorphism of algebras $\mathcal{O}(U_{\phi(x)}) \to (C^\infty(U_x) \otimes_{\mathbf{R}} \mathbf{C}) \otimes_{\mathbf{C}} \mathbf{R})$ that its reduction modulo the maximal ideal $m_R \subset R$ coincides with ϕ^*. In order to complete the proof we notice that $\mathbf{C}^{0|1}$-invariance is equivalent to $\overline{partial}$-closedness, hence we get holomorphic maps $X \to Y$. Therefore we obtained a morphism of functors $Def_M \to Def^\phi$. Since the above construction can be inverted, we see that in fact we have an isomorphism of functors. ■

CHAPTER 5

Operads and algebras over operads

1. Generalities on operads

1.1. Polynomial functors, operads, algebras. Let k be a field of characteristic zero. All vector spaces below will be k-vector spaces unless we say otherwise.

We fix a category $\mathcal{C}$ which is assumed to be k-linear abelian symmetric monoidal and closed under infinite sums and products. We will also assume that it has inner $Hom's$. Our main examples will be the category of k-vector spaces, the category $Vect_k^{\mathbf{Z}}$ of $\mathbf{Z}$-graded vector spaces (with the Koszul rule of signs), and the category of complexes of k-vector spaces.

Suppose we have a collection of representations $F = (F(n))_{n\geq 0}$ of the symmetric groups $S_n, n = 0, 1, ...$ in $\mathcal{C}$ (i.e. we have a sequence of objects $F(n)$ together with an action of the group S_n on $F(n)$ for each n).

Definition 1.1.1. A polynomial functor $F : \mathcal{C} \to \mathcal{C}$ is defined on objects by the formula

$$F(V) = \oplus_{n\geq 0}(F(n) \otimes V^{\otimes n})_{S_n}$$

where for a group H and an H-module W we denote by W_H the space of coinvariants. Functor F is defined on morphisms in an obvious way.

Notice that having a sequence $F(n)$ as above we can define F_I for any finite set I using isomorphisms of I with the standard set $\{1, ..., |I|\}$, where $|I|$ is the cardinality of I. Thus $F_{\{1,...,n\}} = F(n)$. Technically speaking, we consider a functor Φ from the groupoid of finite sets (morphisms are bijections) to the symmetric monoidal category $\mathcal{C}$. Then we set $F_I = \Phi(I)$.

Polynomial functors on $\mathcal{C}$ form a category $\mathcal{PF}$ if we define morphisms between two such functors F and G as a vector space of S_n-intertwiners

$$Hom(F, G) = \prod_{n=0}^{\infty} Hom_{S_n}(F(n), G(n))$$

There is a composition operation $\circ$ on polynomial functors such that $(F \circ G)(V)$ is naturally isomorphic to $F(G(V))$ for any $V \in \mathcal{C}$. We also have a polynomial functor $\mathbf{1}$ such that $\mathbf{1}_1 = 1_{\mathcal{C}}$ and $\mathbf{1}(n) = 0$ for all $n \neq 1$. Here $1_{\mathcal{C}}$ is the unit object in the monoidal category $\mathcal{C}$. It is easy to see that in this way we get a monoidal structure on $\mathcal{PF}$.

Definition 1.1.2. An operad in $\mathcal{C}$ is a monoid in the monoidal category $\mathcal{PF}$. In other words it is a polynomial functor $R \in \mathcal{PF}$ together with morphisms $m : R \circ R \to R$ and $u : \mathbf{1} \to R$ satisfying the associativity and the unit axioms.

More explicitly, an operad is given by a collection of morphisms

$$F(n) \otimes F(k_1) \otimes ... \otimes F(k_n) \to F(k_1 + ... + k_n)$$

$(f, f_1, ..., f_n) \mapsto \gamma(f, f_1, ..., f_n)$ called operadic compositions such that:

a) they are equivariant with respect to the action of the group $S_n \times S_{k_1} \times ... \times S_{k_n}$ on the source object and the group $S_{k_1+...+k_n}$ on the target object;

b) the associativity axiom is satisfied, i.e. two natural compositions $F(n) \otimes F(k_1) \otimes ... \otimes F(k_n) \otimes F(l_{11}) \otimes ... \otimes F(l_{n,l_n}) \to F(\sum_{ij} l_{ij})$ equal to $\gamma(\gamma \otimes id)$ and $\gamma(id \otimes \gamma^{\otimes n})$ coincide.

We are given a morphism $e : 1_{\mathcal{C}} \to F(1)$, which satisfies the unit axiom: $\gamma(e, f) = f, \gamma(f, e, e, ..., e) = f$ for any $f \in F(n)$.

To shorten the notation we will denote the operad (R, m, u) simply by R. An operad R gives rise to a so-called *triple* in the category $\mathcal{C}$. There is the notion of an algebra over a triple in a category. We can use it in order to give a definition of an algebra in $\mathcal{C}$ over the operad R. It is given by an object $V \in \mathcal{C}$ and a morphism $R(V) \to V$ satisfying natural properties of compatibility with the structure of a triple. Equivalently, V is an R-algebra iff there is a morphism of operads $R \to \mathcal{E}nd(V)$, where $\mathcal{E}nd(V)$ is the endomorphism operad of V defined by $(\mathcal{E}nd(V))(n) = \underline{Hom}(V^{\otimes n}, V), n \geq 1$, and $\underline{Hom}$ denotes the inner Hom in $\mathcal{C}$. The unit is given by $id_V \in \mathcal{E}nd(V)(1)$. Actions of the symmetric groups and the operadic compositions are the obvious ones.

The category of R-algebras will be denoted by $R - alg$. There are two adjoint functors $Forget_R : R - alg \to \mathcal{C}$ (forgetful functor) and $Free_R : \mathcal{C} \to R - alg$ such that $Forget_R \circ Free_R = R$.

Definition 1.1.3. For $X \in Ob(\mathcal{C})$ we call $Free_R(X)$ the free R-algebra generated by X.

Abusing notation we will sometimes denote $Free_R(X)$ by $R(X)$. More explicitly, an R-algebra structure on X is given by a collection of linear maps $\gamma_X : R(n) \otimes_{S_n} X^{\otimes n} \to X$, satisfying the associativity condition as well as compatibility with the unit.

There is also a dual notion of *cooperad*. Cooperad is the same as *cotriple* in the category $\mathcal{C}$. The axioms for cooperads are dual to those for operads. In particular, we have a collection of maps

$$F(k_1 + ... + k_n) \to F(n) \otimes F(k_1) \otimes ... \otimes F(k_n)$$

satisfying the coassocitivity property. We leave to the reader to write down explicit diagrams for cooperads.

1.2. Examples of operads. There are operads As, Lie, $Comm$ such that the algebras over them in the category of vector spaces are non-unital associative, Lie and non-unital commutative algebras correspondingly.

We have:

a) $As(n) = k[S_n], n \geq 1$, which is the group algebra of the symmetric group, considered with the right regular action of S_n. Operadic composition is induced by the natural map $S_n \times S_{k_1} \times ... \times S_{k_n} \to S_{k_1+...+k_n}$ such that $\sigma \times \sigma_1 \times ... \times \sigma_n \mapsto \sigma(\sigma_1, ..., \sigma_n)$.

b) $Comm(n) = k$ for all $n \geq 1$, $Comm(0) = 0$. Operadic composition is given by the multiplication in k.

c) $Lie(n) = k[S_n]^{sgn}$ which is the representation of S_n corresponding to the "sign" character.

REMARK 1.2.1. It is customary to describe operads implicitly, by saying what are algebras over them. Intuitively this means that each operadic space $F(n)$ describes "universal" operations in algebras. For example, we can describe the operad of commutative algebras $Comm$ by saying that algebras over this operad are non-unital commutative associative algebras. To define an algebra V over $Comm$ is the same as to define for any $n \geq 1$ a space of linear maps $V^{\otimes n} \to V, v_1 \otimes ... \otimes v_n \mapsto v_1 ... v_n$. Since the actions of the symmetric groups are trivial, we get a commutativity of the multiplication $v_1 \otimes v_2 \mapsto v_1 v_2$.

1.3. Colored operads. There is a generalization of the notion of operad. It is useful in order to describe in operadic terms pairs *(associative algebra A, A-module)*, homomorphisms of algebras over operads, etc.

Let I be set. We consider the category $\mathcal{C}^I$ consisting of families $(V_i)_{i\in I}$ of objects of $\mathcal{C}$.

A polynomial functor $F : \mathcal{C}^I \to \mathcal{C}^I$ is defined by the following formula:

$$(F((V_i)_{i\in I}))_j = \oplus_{a:I\to\mathbf{Z}_{\geq 0}} F_{a,j} \otimes_{\prod_i S_{a(i)}} \otimes_{i\in I}(V_i^{\otimes a(i)})$$

where $a : I \to \mathbf{Z}_+$ is a map with the finite support, and $F_{a,j}$ is a representation in $\mathcal{C}$ of the group $\prod_{i\in I} S_{a(i)}$.

Polynomial functors in $\mathcal{C}^I$ form a monoidal category with the tensor product given by the composition of functors.

DEFINITION 1.3.1. A colored operad is a monoid in this category.

Similarly to the case of usual operads it defines a triple in the category $\mathcal{C}^I$. Therefore we have the notion of an algebra over a colored operad.

There exists a colored operad $\mathcal{OP}$ such that the category of $\mathcal{OP}$-algebras is equivalent to the category of operads.

Namely, let us consider the forgetful functor $Operads \to \mathcal{PF}$. It has a left adjoint functor. Thus we have a triple in $\mathcal{PF}$. As we have noticed before, the category $\mathcal{PF}$ can be described as a category of sequences $(P(n))_{n\geq 0}$ of S_n-modules. Then using the representation theory of symmetric groups, we conclude that the category $\mathcal{PF}$ is equivalent to the category $\mathcal{C}^{I_0}$, where I_0 is the set of all Young diagrams (partitions). Hence a polynomial functor $F : \mathcal{PF} \to \mathcal{PF}$ can be described as a collection $F((m_i), n)$ of the representations of the groups $S_{n,(m_k)} := S_n \times \prod_{k\geq 0}(S_{m_k} \ltimes S_k^{m_k})$, where $\ltimes$ denotes the semidirect product of groups.

Having these data we can express any polynomial functor F on $\mathcal{PF}$ by the formula:

$$(F((U(k))_{k\geq 0}))(n) = \bigoplus_{(m_k)} F((m_k), n) \otimes_{S_{1,(m_k)}} \bigotimes_{k\geq 0} (U(k)^{\otimes m_k})$$

In particular, one has a functor $\mathcal{OP} : \mathcal{PF} \to \mathcal{PF}$, which is the composition of the forgetful functor $Operads \to \mathcal{PF}$ with its adjoint. It gives rise to an I_0-colored operad $\mathcal{OP} = (\mathcal{OP}_{(m_i),n})$. We will describe it explicitly in the subsection devoted to trees.

1.4. Non-linear operads. We remark that operads and algebras over operads can be defined for any symmetric monoidal category $\mathcal{C}$, not necessarily k-linear. In particular, we are going to use operads in the categories of sets, topological spaces, etc.

Namely, an operad in $\mathcal{C}$ is a collection $(F(n))_{n\geq 0}$ of objects in $\mathcal{C}$, each equipped with an S_n-action, as well as composition maps:

$$F(n)\otimes F(k_1)\otimes ...\otimes F(k_n)\to F(k_1+...+k_n)$$

for any $n\geq 0, k_1,...,k_n\geq 0$. Another datum is the unit, which is a morphism $\mathbf{1}_{\mathcal{C}}\to F(1)$. All the data are required to satisfy axioms similar to those of linear operads (see [Ma77]). Analogously one describes colored operads and algebras over operads. Notice that in this framework one cannot speak about polynomial functors and free algebras.

This approach has some advantages and drawbacks (cf. the description of analytic functions in terms of Taylor series versus their description in terms of Taylor coefficients).

1.5. Pseudo-tensor categories and colored operads. The notion of colored operad is essentially equivalent to the notion of pseudo-tensor category discussed before. Pseudo-tensor category with one object is the same as an operad. General pseudo-tensor category is a colored operad with colors given by objects of the category.

If $\mathcal{A}$ is a set, then a pseudo-tensor category is exactly the same as an $\mathcal{A}$-colored operad in the tensor category $\mathcal{V}$.

If we take $\mathcal{V}$ to be the category of sets, and take all sets I (see the definition of a pseudo-tensor category) to be 1-element sets, we obtain the notion of a category with the class of objects equal to $\mathcal{A}$.

Colored operad with one color gives rise to an ordinary operad. A symmetric monoidal category $\mathcal{A}$ produces the colored operad with $P_I((X_i),Y)=Hom_{\mathcal{A}}(\otimes_i X_i, Y)$.

The notion of pseudo-tensor category admits a generalization to the case when no action of symmetric group is assumed. This means that we consider *sequences* of objects instead of *families* (see [So99]). The new notion generalizes monoidal categories. In terms of the next subsection this would mean that one uses planar trees instead of all trees. One can make one step further generalizing braided categories. This leads to colored braided operads (or pseudo-braided categories). In this case trees in $\mathbf{R}^3$ should be used.

2. Trees

In this book we use graphs and trees. For different purposes we will need different classes of graphs and trees. We prefer to define each class individually. In the case of operads, trees are used as tools for visualization of operadic compositions.

DEFINITION 2.0.1. A tree T is defined by the following data:

1) a finite set $V(T)$ whose elements are called vertices;

2) a distinguished element $root_T\in V(T)$ called root vertex;

3) subsets $V_i(T)$ and $V_t(T)$ of $V(T)\setminus\{root_T\}$ called the set of internal vertices and the set of tails respectively. Their elements are called internal and tail vertices respectively;

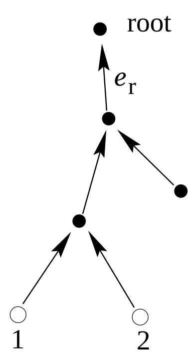
4) a map $N=N_T: V(T)\to V(T)$.

These data are required to satisfy the following properties:
a) $V(T) = \{root_T\} \sqcup V_i(T) \sqcup V_t(T)$;
b) $N_T(root_T) = root_T$, and $N_T^k(v) = root_T$ for all $v \in V(T)$ and $k \gg 1$;
c) $N_T(V(T)) \cap V_t(T) = \emptyset$;
d) there exists a unique vertex $v \in V(T), v \neq root_T$ such that $N_T(v) = root_T$.

We denote by $|v|$ the *valency* of a vertex v, which we understand as the cardinality of the set $N_T^{-1}(v)$.

We call the pairs $(v, N(v))$ *edges* in the case if $v \neq root_T$. If both elements of the pair belong to $V_i(T)$ we call the corresponding edge *internal.* The only edge e_r defined by the condition d) above is called the *root edge*. All edges of the type $(v, N(v)), v \in V_t(T)$ are called *tail edges*. We use the notation $E_i(T)$ and $E_t(T)$ for the sets of internal and tail edges respectively. We have a decomposition of the set of all edges $E(T) = E_i(T) \sqcup (E_t(T) \cup \{e_r\})$. There is a unique tree T_e such that $|V_t(T_e)| = 1$ and $|V_i(T_e)| = 0$. It has the only tail edge which is also the root edge.

A *numbered* tree with n tails is by definition a tree T together with a bijection of sets $\{1, ..., n\} \to V_t(T)$. We can picture trees as follows



Numbered tree,
non-numbered vertices are black

Let R be an operad. Any tree T gives a natural way to compose elements of R, $comp_T : \otimes_{i \in V_i(T)} R(N^{-1}(v)) \to R(V_t(T))$.

Let us return to the colored operad $\mathcal{OP}$ and give its description using the language of trees.

Namely, $\mathcal{OP}((m_i), n)$ is a k-vector space generated by the isomorphism classes of trees T such that:

a) T has n tails numbered from 1 to n;

b) T has $\sum_i m_i$ internal vertices all numbered in such a way that first m_0 vertices have valency 0, and they are numbered from 1 to m_0, next m_1 internal vertices have valency 1, and they are numbered from 1 to m_1, and so on;

c) for every internal vertex $v \in V_i(T)$ the set of incoming edges $N_T^{-1}(v)$ is also numbered.

An action of the group $S_{(m_k),n}$ is defined naturally: the factor S_n permutes numbered tails, the factor S_{m_k} permutes numbered internal vertices and the factor $S_k^{m_k}$ permutes their incoming edges numbered from 1 to k.

The composition is given by the procedure of inserting of a tree into an internal vertex of another one. The new numeration is clear. We leave these details as well as the proof of the following proposition to the reader.

PROPOSITION 2.0.2. *The category of $\mathcal{OP}$-algebras is equivalent to the category of k-linear operads.*

Let F be a polynomial functor on $\mathcal{C}$. Let us consider a category $\mathcal{C}_F$ objects of which are pairs $(V, \phi : F(V) \to V)$ where V is an object of $\mathcal{C}$ and ϕ is a morphism in $\mathcal{C}$. Morphisms of pairs are defined in the natural way.

PROPOSITION 2.0.3. *The category $\mathcal{C}_F$ is equivalent to the category of $Free_{\mathcal{OP}}(F)$-algebras.*

*Proof.*Exercise. ■

We call $P = Free_{\mathcal{OP}}(F)$ the *free operad* generated by F.

Components $P(n)$ of the functor P can be defined explicitly as follows.

Let $Tree(n)$ denotes the groupoid of numbered trees with n tails, $|Tree(n)|$ denotes the set of classes of isomorphisms of these trees . We denote the class of isomorphism of T by $[T]$. Then we have

$$P(n) = Free_{\mathcal{OP}}(F)(n) = \oplus_{[T]\in|Tree(n)|}(\otimes_{v\in V_i(T)}F(N^{-1}(v)))_{AutT}$$

3. Resolutions of operads

3.1. Topological motivation. Let R be an operad over a field k. This means that R is an operad in the tensor category of k-vector spaces. The aim of this section is to discuss the notion of *resolution of* R. In particular we will construct canonically a dg-operad P_R over k, which is free as a graded operad, and a quasi-isomorphism $P_R \to R$. In this subsection we will assume that R is non-trivial, which means that the unit operation from $R(1)$ is not equal to zero.

As a motivation we start with some topological construction (cf. [BV73]).

Let $O = (O(n))_{n\geq 0}$ be a topological operad (i.e. all $O(n)$ are S_n-topological spaces and all operadic morphisms are continuous). We describe (following [BV73]) a construction of topological operad $B(O) = (B(O)(n))_{n\geq 0}$ together with a morphism of topological operads $B(O) \to O$ which is homotopy equivalence.

To simplify the exposition we assume that S_n acts freely on O_n for all n.

Each space $B(O)(n)$ will be the quotient of

$$\widehat{B(O)}(n) = \coprod_{[T],T\in Tree(n)} ([0,+\infty]^{E_i(T)} \times \prod_{v\in V_i(T)} O(N^{-1}(v)))/AutT$$

by the relations described such as follows.

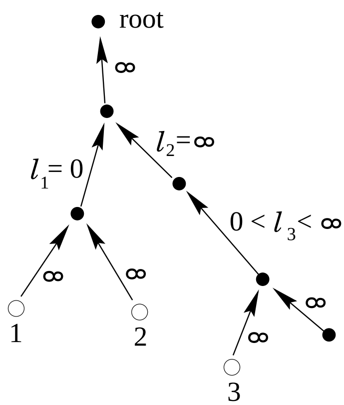
Let us consider elements of $\widehat{B(O)}(n)$ as numbered trees with elements of O attached to the internal vertices, and the *length* $l(e) \in [0,+\infty]$ attached to every edge e. We require that all external edges (i.e. root edge and the tail edges) have lengths $+\infty$.

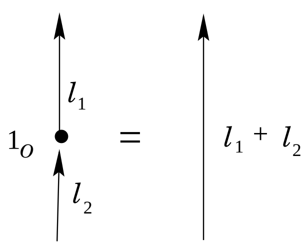
We impose two type of relations.

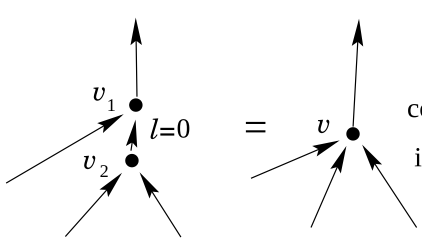
1) We can delete every vertex v of valency 1 if it contains the unit of the operad, replacing it and the attached two edges of lengths $l_i, i = 1, 2$ by the edge with the length $l_1 + l_2$. We use here the usual assumption: $l + \infty = \infty + l = \infty$.

2) We can contract every internal edge $e = (v_1, v_2), v_2 = N(v_1)$ of the length 0 and compose in O the operations attached to $v_i, i = 1, 2$.

We depict the trees and relations below.







Let us describe how $B(O)$ can act naturally on a topological space.

Let X be a topological space, Y a topological subspace, and $g^t : X \to X, t \in [0, +\infty)$ a 1-parametric semigroup of continuous maps acting on X. We assume that for any $x \in X$ the limit

$$x_\infty = \lim_{t\to\infty} g^t x$$

exists and belongs to Y. We use the notation g^∞ for the corresponding continuous map $X \to Y, x \mapsto x_\infty$. We have also a continuous map $[0, +\infty] \times X \to X, (t, x) \mapsto g^t(x)$.

Suppose that a topological operad O acts on X, i.e. we are given continuous maps $O(n) \times X^n \to X, n \geq 0$, satisfying the usual properties. We can construct an action of $B(O)$ on Y as follows. Let $\gamma \in O(n), t, t_i \in \mathbf{R}_+ \cup \{+\infty\}, x_i \in X, 1 \leq i \leq n$. Then we assign to these data the point $x = g^t\gamma(g^{t_1}x_1, ..., g^{t_n}x_n)$ of X. We define the composition of such operations in the natural way.

We can interpret the parameters t, t_i above as lengths of edges of trees . Putting $t = +\infty$ we obtain an action of $B(O)$ on a the homotopy retract Y.

4. Resolutions of linear operads

4.1. Filtered resolutions of algebras. Let $\mathcal{C}$ be a k-linear abelian tensor category, and $V \in Ob(\mathcal{C})$ be a filtered object, i.e. $V = \bigcup_{n\geq 0} V_{\leq n}$, where $\{0\} = V_{\leq 0} \subset V_{\leq 1} \subset ....$ is an increasing filtration, and R be an operad in $\mathcal{C}$. Viewing R as a polynomial functor we see that R transforms monomorphisms into monomorphisms. In particular $R(V_{\leq n}) \subset R(V)$.

Since $\mathcal{C}$ is an abelian tensor category we can speak about derivations of the free R-algebra $R(V) = Free_R(V)$ as of a Lie algebra in the tensor category $Ind(\mathcal{C})$ of ind-objects. More precisely, for any R-algebra A in $\mathcal{C}$ and any commutative nilpotent algebra m in $\mathcal{C}$ we consider an R-algebra $A \otimes (1_\mathcal{C} \oplus m)$. Let us consider a set of automorphisms $f : A \otimes (1_\mathcal{C} \oplus m) \to A \otimes (1_\mathcal{C} \oplus m)$ such that

a) f is a morphism of $1_\mathcal{C} \oplus m$-modules;

b) f is a morphism of R-algebras;

c) reduction of f modulo m is equal to id_A.

It is easy to see that in this way we get a functor from the category of commutative nilpotent algebras in $\mathcal{C}$ to the category of groups.

Exercise 4.1.1. Prove that this functor is represented by an object of $Ind(\mathcal{C})$.

In other words we have a group object in the category of formal schemes in $\mathcal{C}$ (see Appendix). On can prove that in fact it is a formal manifold. It has a marked point, which is the unit of the formal group. The tangent space at the unit is a Lie algebra in $Ind(\mathcal{C})$. It is denoted by $\underline{Der}(A)$ and called the Lie algebra of derivations of A.

Exercise 4.1.2. Prove the following isomorphism of objects:

$$\underline{Der}(Free_R(V)) \simeq \underline{\mathrm{Hom}}(V, R(V)).$$

(Hint: use the fact that $R(V) = Forget_R(Free_R(V))$).

From now on we will assume that $\mathcal{C} = Vect_k$. Then we can speak about elements of $\underline{Der}(A)$. Let us return to the case $A = Free_R(V)$, where V is a filtered vector space.

Definition 4.1.3. We say that $D \in \underline{Der}(Free_R(V))$ lowers the order of filtration by m if $D(V_{\leq n}) \subset R(V_{\leq n-m})$.

We will be interested in the case $m = 1$. Then D transforms the filtration $V_{\leq 1} \subset V_{\leq 2} \subset ....$ into the filtration $\{0\} = R(V_{\leq 0}) \subset R(V_{\leq 1}) \subset .....$

Let now $A = (Free_R(V), d)$ be a dg-algebra over R such that the differential d is a derivation which lowers the order of filtration by 1. In this case we say that A is a *cofibrant* R-algebra.

PROPOSITION 4.1.4. *For any dg-algebra B over R there exists a cofibrant R-algebra A which is quasi-isomorphic to B. If A' is another such a cofibrant R-algebra then there is a cofibrant R-algebra A'' which is quasi-isomorphic to both A and A'.*

Proof. We need to find a filtered V such that $A = Free_R(V)$. We set $V_{\leq 1} = d(Forget_R(B)), V_{\leq 2} = Forget_R(B)$. Then $d(V_{\leq 1}) = 0, d(V_{\leq 2}) \subset R(V_{\leq 1})$. There is natural morphism of dg-algebras $R(V_{\leq 2}) \to B$, which is an epimorphism on cohomology. Let us split $Ker(d_{|R(V_{\leq 2})})$ as a direct sum of the image U of the space $Ker(H^\bullet(R(V_{\leq 2}) \to H^\bullet(B))$ and some $\mathbf{Z}$-graded vector space U_1. Set $V_{\leq 3} = U[1] \oplus V_{\leq 2}$. The differential $d_{V_{\leq 3}}$ is defined as a sum of the old differential $d_{V_{\leq 2}}$ and $d_{U[1]}$ induced by the splitting. Then $d_{V_{\leq 3}}(V_{\leq 3}) \subset R(V_{\leq 2})$ by construction. Notice that elements of $U[1]$ are cohomologically trivial. Then we can add generators corresponding to these classes. Thus we have a morphism of dg-algebras $R(V_{\leq 3}) \to B$. Repeating the procedure we construct by induction all $V_{\leq n}, n \geq 1$. It is easy to see that $A = Free_R(V)$ gives the desired cofibrant R-algebra. The uniqueness is proved similarly, if one uses the fact that all cofibrant R-algebras are isomorphic as $\mathbf{Z}$-graded R-algebras (i.e. we forget about differentials). ■

COROLLARY 4.1.5. *If A is a cofibrant R-algebra then $\underline{Der}(A)$ is a DGLA. For a quasi-isomorphic cofibrant R-algebra B one has a quasi-isomorphism of DGLAs $\underline{Der}(A) \simeq \underline{Der}(B)$.*

Proof ??? ■

4.2. Boardman-Vogt resolution of a linear operad. Considerations of the previous subsection can be applied to algebras over colored operads as well. In particular we can speak about filtered resolutions of operads (they are algebras over the colored operad $\mathcal{OP}$).

Let us return to the construction of the resolution $P_R \to R$. In order to describe a dg-operad P_R we need a special class of trees. More precisely, for every $n \geq 0$ we introduce a groupoid $\mathcal{T}(n)$ of *marked trees* with n tails. An object of $\mathcal{T}(n)$ is a numbered tree $T \in Tree(n)$ and a map to a 3-element set $l_T : E(T) \to \{0, finite, +\infty\}$ such that $l_T(\{e_r\} \cup E_t(T)) = \{+\infty\}$. Notice that in the case of topological operads the component $\widehat{B(O)}(n)$ is stratified naturally with the strata labeled by equivalence classes $|\mathcal{T}(n)|$ of marked trees. The label of an edge e of the corresponding marked tree is 0 if $l(e) = 0$, is $finite$ if $l(e) \in (0, +\infty)$ and is $+\infty$ if $l(e) = +\infty$. According to this description we call them $zero$-$edges$, $finite$ edges or $infinite$ edges respectively. We denote these sets of edges by E_{zero}, E_{finite} and $E_{infinite}$ correspondingly.

We will give three different descriptions of the operad $P = P_R$ as a graded operad. Then we define a differential.

Description 1.

Let

$$\bar{P}(n) = \bigoplus_{[T], T \in \mathcal{T}(n)} (\otimes_{v \in V(T)} R(N^{-1}(v))[J_T])_{AutT}$$

where $AutT$ is the group of automorphisms of the tree T, $J_T = l_T^{-1}(0, +\infty)$, and for any graded vector space W and a finite set J we use the notation $W[J] = W \otimes k[1]^{\otimes J}$ (shift of the grading by J).

Note that the dimension of the corresponding stratum of $\mathcal{T}(n)$ is the cardinality of the set J_T= the number of finite edges.

Then $(\bar{P}(n))_{n \geq 0}$ evidently form a graded operad $\bar{P}$. It is a k-linear analog of the operad $\widehat{BO}$.

The operad $\bar{P}$ contains a subspace I generated by the following relations

1) if the length of an edge (w, v) is 0 we contract it and make the composition in R of the operations attached to w and v (cf. description for $B(O)$);

2) for any vertex v of valency 1 with the unit $1_R \in R_1$ attached , we replace it by 0 if at least one attached edge is finite. If they are both infinite, we remove the vertex and two edges, replacing them by an infinite edge.

One can check easily that I is a graded ideal in $\bar{P}$. We denote by P the quotient operad $\bar{P}/I$.

Description 2.

We define $P(n)$ by the same formula as above, but making a summation over the trees without edges of zero length. We also drop the relation 1) from the list of imposed relations (there are no edges with $l = 0$).

Description 3.

We define an operad R' such as follows:

$R'(n) = R(n)$ for $n \neq 1$, $R'(1)$= a complement to $k \cdot Id$ in $R(1)$.

Then we define $P(n)$ as in Description 2, but using R' instead of R and dropping both relations 1) and 2).

It is clear that this description defines a free graded operad.

Equivalently, it can be described as a free graded operad P such that

$$P = Free(Cofree'(R'[1]))[-1]$$

Here $Cofree(L)$ means a dg-cooperad generated by L which is cofree as a graded co-operad, and $'$ denotes the procedure of taking a (non-canonical) complement to the subspace generated by the unit (or counit in the case of a co-operad) as described above in the case of R.

In this description the generators of P correspond to such trees T in $\mathcal{T} = (\mathcal{T}(n))_{n \geq 0}$ that every T has at least one internal vertex, all internal edges are finite and there are no zero-edges in T.

PROPOSITION 4.2.1. *All three descriptions give rise to isomorphic graded free operads over k.*

Proof. Exercise. ■

We can make $\bar{P}$ into a dg-operad introducing a differential $d_{\bar{P}}$. We use the Description 1 for this purpose.

The differential d_P is naturally decomposed into the sum of two differentials:

$d_P = \tilde{d}_R + d_{\mathcal{T}}$ where

a) the differential $\tilde{d}_R$ arises from the differential d_R in R;

b) the differential $d_{\mathcal{T}}$ arising from the stratification of $\mathcal{T}(n)$: it either contracts a finite edge or makes it into an infinite edge .

To be more precise, let us consider the following object Δ in $\mathcal{C} = Vect_k^{\mathbf{Z}}$: $\Delta^{-1} = 1_{\mathcal{C}}, \Delta^0 = 1_{\mathcal{C}} \oplus 1_{\mathcal{C}}$ where $1_{\mathcal{C}}$ is the unit object in the monoidal category $\mathcal{C}$. Then Δ can be made into a chain complex of the CW complex $[0,1] = \{0\} \cup (0, +\infty) \cup \{+\infty\}$.

We see that as a graded space our $\bar{P}(n)$ is given by the formula

$$\bar{P}(n) = \oplus_{[T], T \in \mathcal{T}(n)} (\otimes_{v \in V_i(T)} R(N^{-1}(v)) \otimes \Delta^{\otimes E_i(T)})_{AutT}$$

Since we have here a tensor product of complexes, we get the corresponding differential $d_{\bar{P}}$ in $\bar{P}$.

PROPOSITION 4.2.2. *The ideal I is preserved by $d_{\bar{P}}$.*

Proof. Straightforward computation. ■

Therefore $P = P_R$ is a dg-operad which is free as a graded operad.

There is a natural morphism of dg-operads $\phi : P \to R$. In terms of the Description 2 it can be defined such as follows:

ϕ sends to zero all generators of P corresponding to trees with at least one finite edge. Let $T \in P$ be a tree with all infinite edges. Then T gives rise to a natural rule of composing in R elements of $R(N^{-1}(v))$ assigned to the vertices of T. We define $\phi(T) \in R$ as the result of this composition.

It is easy to check that ϕ is a well-defined morphism of dg-operad.

PROPOSITION 4.2.3. *The morphism ϕ is a quasi-isomorphism of dg-operads.*

Proof. The proof follows from the spectral sequence arising from the natural stratification of $\mathcal{T}$. To say it differently, let us consider the tautological embedding ψ of R into P. Then ψ is a right inverse to ϕ. It gives a splitting of P into the sum $P = \psi(R) \oplus P^{(0)}$. Here $P^{(0)}$ is spanned by the operations corresponding to trees with finite edges only. Such a tree can be contracted to a point which means that $P^{(0)}$ is contractible as a complex. Hence ϕ defines a quasi-isomorphism of complexes and dg-operads. ■

We are going to call the resolution $P_R \to R$ the *Boardman-Vogt resolution* of the operad R, or simply BV-resolution of R (notation $BV(R)$).

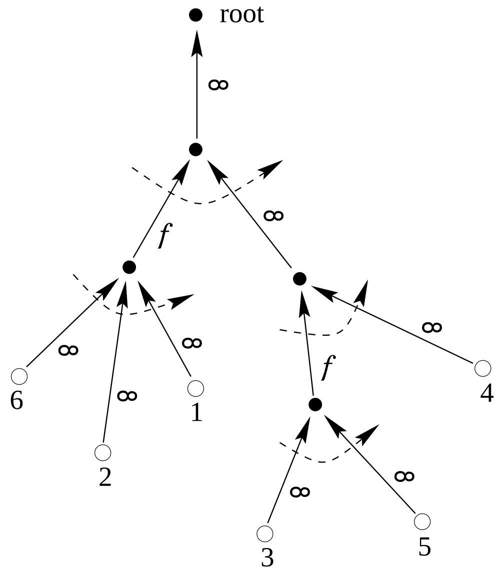
4.3. Example: BV-resolution of an associative operad. Let us discuss an example when R is the operad of associative algebras without the unit. We denote it by As. Then for any $n \geq 1$ we have: $As(n)$ is isomorphic to the regular representation of the symmetric group S_n.

In this case the complex $P(n)$ from the previous subsection can be identified with the chain complex of the CW-complex $K_n, n \geq 2$ described below.

The cells of K_n are parametrized by *planar trees* with an additional structure on edges. By a planar tree here we understand a numbered tree T such that for any $v \in V_i(T)$ the cardinality of $N^{-1}(v)$ is at least 2 and this set is completely ordered. The additional structure is a map $E_i(T) \to \{finite, infinite\}$. We call an edge finite or infinite according to its image under this map. Dimension of the cell is equal to the number of finite edges of the corresponding planar tree.

We can either contract a finite edge or make it infinite. This defines an incidence relation on the set of cells.

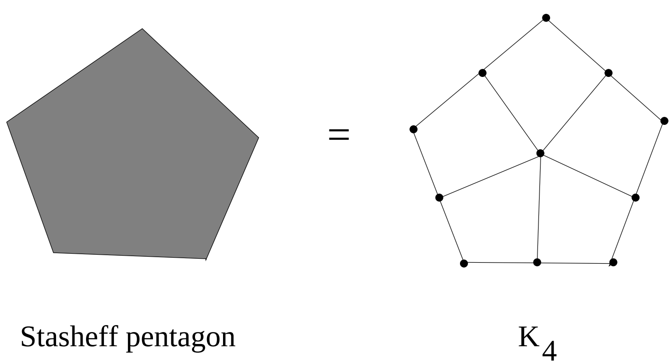
We can picture planar trees as follows



Here the dashed lines show the complete orders on set of incoming edges. We will not show them on other figures in the text. Instead, we will tacitly assume that for a given vertex the incoming edges are completely ordered from the left to the right.

In this way we obtain simplicial subdivisions of the Stasheff polyhedra.

We depict the case $n = 4$ below



4.4. Cofibrant BV-resolution. One problem with the Boardman-Vogt resolution is that it is not cofibrant. There is a slight modification of it, which is cofibrant. In order to do this we introduce for each integer $n \geq 0$ a groupoid $\mathcal{T}^c(n)$ of trees which we call Christmas trees. It is a slight modification of the groupoid $\mathcal{T}(n)$ introduced in Section 3.2. In addition to the lengths of edges described there, we also allow a new type of vertices, which we call *lights*. They are internal vertices of the (incoming) valency 1. To each light v we assign a real number $b(v) \in [0, +\infty]$ called the *brightness* of v. Now the terminology becomes clear. Then repeat the construction of the resolution $BV(R)$, with the following modifications:

a) We do not replace by 0 a vertex of valency one, with the 1_R inserted. Instead, we keep all such vertices and, in addition, we assign to such a vertex a new number, namely, the brightness of the vertex. Then each vertex becomes a light.

b) When we construct the differential we use the following rules:

b1) contraction of an edge between two lights $(v_1, b(v_1))$ and $(v_2, b(v_2))$ leads to creation of a new light $(v, b(v)) = b(v_1) + b(v_2))$;

b2) contraction of an edge between the light and an internal vertex removes the light from the set of vertices without any other changes.

In this way we obtain a cofibrant resolution of R denoted by $BV^c(R)$ (cofibrant BV-resolution).

5. Resolutions of classical operads

In this subsection we describe resolutions of the classical operads As (associative non-unital algebras), Lie (Lie algebras) and $Comm$ (non-unital associative commutative algebras).

5.1. Generators. a) The resolution of As which we are going to construct is called $\mathcal{A}_\infty$-operad. Algebras over this operad are called A_∞-algebras. They will be studied in detail in Chapter 6.

At the level of generators the operad $\mathcal{A}_\infty$ is given by the polynomial functor $G(V) = \oplus_{n\geq 2} V^{\otimes n} = (\oplus_{n \geq 2}(V[1])^{\otimes n})[-2]$, where $V \in Vect_k^{\mathbf{Z}}$.

b) Resolution of the operad Lie is called $\mathcal{L}_\infty$-operad. Algebras over this operad are L_∞-algebras discussed in Chapter 3. At the level of generators it is given by a polynomial functor $G(V) = \oplus_{n\geq 2} \bigwedge^n V[n-2] = (\oplus_{n\geq 2} S^n(V[1]))[-2]$.

c) Resolution of the operad $Comm$ is denoted by $\mathcal{C}_\infty$. At the level of generators it is given by the polynomial functor $G(V) = (\oplus_{n\geq 2} Lie^n(V[1]))[-2]$, where $Lie^n(U)$ denotes the vector space spanned by homogeneous components of degree n in the free Lie algebra generated by U.

We can write down all three polynomial functors in a uniform way, using the fact that all three classical operads are quadratic, i.e. they can be written as quotients of the free operad by the ideal generated by quadratic relations. For a quadratic operad R one has the notion of quadratic dual operad $R^!$ introduced in [GiKa94]. In particular, the operad As is dual to itself, while Lie and $Comm$ are dual to each other. Let R be either of three classical operads. Then the polynomial functor describing generators of the free resolution of R can be written as

$$G_R(V) = (\oplus_{n\geq 2}[-2] \circ (R^!(n))^* \circ [1])(V).$$

Here $[l]$ denotes the functor of shifting by $l \in \mathbf{Z}$, and $(R^!(n))^*$ is considered as an S_n-module dual to $R^!(n)$. We are going to denote the nth summand (considered as a polynomial functor) by $G_R(n)$.

5.2. Differential. Notice that $(R^!)^* = ((R^!(n))^*)_{n\geq 2}$ is a cooperad. Then we have a cocomposition (let us call it coproduct for short)

$\delta : (R^!(n))^* \to \oplus_{m_1+m_2=n+1}(R^!(m_1))^* \otimes (R^!(m_2))^*$. This formula gives us a differential d on generators.

Let us denote by $x_{T,G_R(n)}$ a generator which can be depicted as a tree T with n tails and the only internal vertex v such that $G_R(n)$ is inserted in the vertex. Then $d(x_{T,G_R(n)})=\sum_{T_1\to T} y_{T_1}$, where the sum is taken over all trees T_1 which are obtained from T by removing v, creating new vertices v_1, v_2 and new edge e with the endpoints v_1 and v_2 (we denote this by $T_1 \to T$), and y_{T_1} is a polynomial functor which has $G_R(m_1)$ inserted in v_1 and $G_R(m_2)$ inserted in the vertex v_2 (notice that insertion of a new edge splits the set of n tails into two subsets consisting of m_1 and m_2 elements, such that $m_1+m_2=n+1$). There are also appropriate signs in the formula.

We can recast the above formula in a different way. In order to do this we observe that in all three cases the cooperad $\Phi=(R^!)^*$ is non-counital, i.e. $\Phi(1)=0$ (since $R(1)=0$ in all three cases, i.e. these operads are non-unital). We can consider Φ as a polynomial functor. Then we can construct a free non-unital operad $Free([-1]\circ\Phi)$. To be pedantic, one should define a colored operad $\mathcal{OP}^{nu}$ such that $\mathcal{OP}^{nu}$-algebras are non-unital operads (at this time it suffices to say that for a non-unital operad R one has $R(1)=0$). After that we set $Free(F)=Free_{\mathcal{OP}^{nu}}(F)$ by definition. Then $Free([-1]\circ\Phi)$ is a dg-operad with the differential given as above. In order to write down proper signs in the formula for the differential we need few more notation.

For a finite set I we denote by $[-I]$ (shift by $-I$) the functor of tensoring with $H_{BM}(\mathbf{R}^I)$, which is the Borel-Moore homology of $\mathbf{R}^I$. Similarly, we denote by $[I]$ (shift by I) the functor of tensoring with $H^{BM}(\mathbf{R}^I)=(H_{BM})^*(\mathbf{R}^I)$, which is the Borel-Moore cohomology of $\mathbf{R}^I$. Then $F=Free([-1]\circ\Phi)=\oplus_T F_T$, where $F_T=[V_i(T)]\circ(\bigotimes_{v\in V_i(T)}\Phi(N^{-1}(v)))$.

Notice that if $T_1\to T$ then one has (in the previous notation) the coproduct map $\delta_{T_1,T}:\Phi(N^{-1}(v))\to\Phi(N^{-1}(v_1))\otimes\Phi(N^{-1}(v_2))$. Then $d_T=d(F_T)=\sum_{T_1\to T}\delta_{T_1,T}\circ F_T$, and $d=\sum_T d_T$.

Let pt denotes a chosen 1-element set. Then the differential has to be a morphism of polynomial functors $d:[-pt]\circ F\to F$. In order to define the differential it suffices to choose an isomorphism of functors $[-pt]\circ[-V_i(T)]\simeq[-V_i(T_1)]$. This means that we need to identify the vertex $v\in V_i(T)$ with one of the new vertices v_1 or v_2 of T_1. There is no canonical choice for such an identification. Two possible choices lead to isomorphic complexes. An isomorphism of the corresponding dg-operads is achieved by multiplication of each component d_T by $(-1)^{|V_i(T)|}$.

The following result is an immediate corollary of our construction.

Proposition 5.2.1. *In all three cases (in fact for any non-counital cooperad) we have* $d^2=0$.

Next thing is to check whether there is a quasi-isomorphism $(F,d)\to R$. This is done in each case individually.

Let us sketch a construction of a morphism $\pi: F\to R$ of dg-operads.

1) One observes that in all three cases one has an isomorphism of S_2-modules:

$$[-2]\circ(R^!(2))^*\circ[1]\simeq R(2).$$

2) All three classical operads quadratic (i.e. they are generated by $R(2)$ with the relations in $R(3)$). For example As is generated by a single generator m_2 (algebra product) with the relation $m_2(m_2\otimes id)=m_2(id\otimes m_2)$.

3) Let F^0 be a summand in F (as an operad in the category of graded vector spaces) which has degree zero. Then F^0 is a free operad generated by $R(2)$, since all other components of F have negative degrees. This gives a morphism of operads $\pi : F \to R$ such that all components of negative degrees are killed by π.

4) The morphism π is compatible with the differential d, i.e. $\pi \circ d = 0$. The compatibility condition is equivalent to the quadratic relations which define R (for example for $R = As$ it is exactly the associativity condition for m_2).

Next question is: why π is a quasi-isomorphism?

There is no universal answer to this question. Typically, one reduces the proof to the computation of the homology of some standard chain complex $K^\bullet$ of geometric origin such that $H^{>0}(K^\bullet) = 0$ and $H^0(K^\bullet)$ is 1-dimensional. For example, in the case of the operad As the corresponding complex is described such as follows.

Let us consider the set of isomorphism classes of all planar trees T with n tails ($n \geq 2$ is fixed) such that $|N^{-1}(v)| \geq 2$ for all internal vertices v. Then we introduce a structure of complex on a $\mathbf{Z}$-graded k-vector space $B^\bullet = \oplus_T k[-V_i(T)]$ by inserting an extra edge, similarly to the definition of d given above.

Exercise 5.2.2. Prove that $B^\bullet$ has trivial cohomology in positive degrees and 1-dimensional cohomology in degree zero. (Hint: $B^\bullet$ is isomorphic to the chain complex of the Stasheff polyhedron K_n).

The case of the operad Lie is more complicated. One can identify $Lie(n)$ with the vector space $H_\bullet(\mathbf{C}^n - diag)$ of configurations of different n points in $\mathbf{C}^n$ modulo shifts, and then use Hodge theory. It would be nice to have a uniform explanation of the quasi-isomorphisms for all classical operads.

6. Deformation theory of algebras over operads

6.1. Statement of the problem and two approaches. Let R be an operad in $Vect_k$, V be an R-algebra. Then we can define a "naive" deformation functor $Def^V : Artin_k \to Sets$. Namely for an Artin k-algebra (C, m_C) we define $Def^V(C)$ to be the set of isomorphism classes of R-algebras B which are also C-modules, such that reduction of B modulo the maximal ideal m_C is isomorphic to V. The problem is to find a formal pointed manifold $\mathcal{M}$ such that the deformation functor $Def_{\mathcal{M}}$ is isomorphic to Def^V.

One has two different approaches to this problem.

First approach

Let $R(E) \to V$ be a cofibrant resolution, where E is a filtered $\mathbf{Z}$-graded vector space. Then we have a DGLA $g_{R(E)} = \underline{Der}(R(E))$. This DGLA gives rise to a deformation functor $Def_{g_{R(E)}}$.

Second approach

Let $P = P_R \to R$ be a resolution of the operad R. For example, we can take a cofibrant resolution (it exists according to Section 3.2). Thus P is a dg-operad, which is free as a graded operad, and the surjective morphism of dg-operads $P \to R$ is a quasi-isomorphism (we endow R with the trivial differential). Then an R-algebra V becomes a dg-algebra over a dg-operad P. Let us forget differentials for a moment. Since P is a free as a graded operad, we can write $P = Free_{\mathcal{OP}}(F)$, where F is a polynomial functor is the category $Vect_k^{\mathbf{Z}}$. Then, instead of studying deformations of V as an R-algebra, we can study deformations of V as an algebra over the graded operad P. In the next subsection we will define a formal pointed dg-manifold $\mathcal{M} = \mathcal{M}(P, V)$ controlling this deformation theory. It gives rise to the deformation functor $Def_{\mathcal{M}}$.

We are interested in the following conjecture.

CONJECTURE 6.1.1. Functors Def^V, $Def_{g_{R(E)}}$ and $Def_{\mathcal{M}}$ are isomorphic to each other.

6.2. Deformations and free operads. Let F be a polynomial functor, $P = Free_{\mathcal{OP}}(F)$ be the corresponding free operad. Let g_P be the Lie algebra (in the symmetric monoidal category $\mathcal{C}$) of derivations of the operad P. Then, as an object of $\mathcal{C}$:

$$g_P = \prod_{n \geq 0} \underline{Hom}(F(n), P(n))^{S_n}$$

where W^H denotes the space of H-invariants of an H-module W and $\underline{Hom}$ denotes the inner Hom in $\mathcal{C}$. This follows from the fact that $Hom_{\mathcal{PF}}(F, Forget_{\mathcal{OP}}(G)) = Hom_{\mathcal{OP}-alg}(Free_{\mathcal{OP}}(F), G)$.

From now on we suppose that $\mathcal{C}$ is the category $Vect_k^{\mathbf{Z}}$ of $\mathbf{Z}$-graded vector spaces. Then g_P is a graded Lie algebra with the graded components g_P^n.

DEFINITION 6.2.1. A structure of differential-graded operad on P which is free as a graded operad is given by an element $d_P \in g_P^1$ such that $[d_P, d_P] = 0$.

The definition means that P can be considered as an operad in the symmetric monoidal category of complexes, and it is free as an operad in the category $Vect_k^{\mathbf{Z}}$. Sometimes we will denote the corresponding operad in the category of complexes by $\widehat{P}$.

One of our purposes will be to use $\widehat{P}$ for constructing resolutions of dg-operads, and subsequently the deformation theory of algebras over them.

DEFINITION 6.2.2. A dg-algebra over (P, d_P) (or simply over P) is an algebra over $\widehat{P}$ in the category of complexes.

Notice that the deformation theory of the pair (P, d_P) is the same as the deformation theory of d_P (since P is free and therefore rigid). Then we can define the deformation theory of the dg-operad (P, d_P) axiomatically in the following way.

DEFINITION 6.2.3. The formal pointed dg-manifold associated with the differential-graded Lie algebra $(g_P, [d_P, \bullet])$ controls the deformation theory of (P, d_P).

Now we are going to describe a formal pointed dg-manifold controlling the deformation theory of dg-algebras over P (i.e. $\widehat{P}$-algebras). Before doing that we recall to the reader that when speaking about points of $\mathbf{Z}$-graded manifolds we always mean Λ-points, where Λ is a nilpotent commutative algebra (or commutative Artin algebra).

Let V be a P-algebra. We have the following graded vector space

$$\mathcal{M} = \mathcal{M}(P, V) = (\underline{Hom}(V, V))[1] \oplus \underline{Hom}(F(V), V)$$

We denote by $\mathcal{M}^n, n \in \mathbf{Z}$ the graded components of $\mathcal{M}$.

The structure of a complex on the graded vector space V and an action of P on V define a point $(d_V, \rho) \in \mathcal{M}^0 = Hom_{Vect_k^{\mathbf{Z}}}(k, \mathcal{M})$. We consider here d_V and ρ as morphisms of graded vector spaces. The equation $d_V^2 = 0$ and the condition of compatibility of d_V and ρ can be written in the form $d_{\mathcal{M}}(d_V, \rho) = 0$, where $d_{\mathcal{M}}(d_V, \rho) = (d_V^2, \xi(d_V, \rho)) \in \mathcal{M}^1$, for some $\xi(d_V, \rho) \in \underline{Hom}(F(V), V)$. It is easy to see that the assignment $(d_V, \rho) \mapsto d_{\mathcal{M}}(d_V, \rho)$ defines an odd vector field $d_{\mathcal{M}}$ on the infinite-dimensional graded manifold $\mathcal{M}$. A zero of this vector field corresponds to a structure of complex on V together with a compatible structure of a dg-algebra over P. This gives a bijection between the set of zeros and the set of such structures.

It is easy to check that $[d_{\mathcal{M}}, d_{\mathcal{M}}] = 0$. Therefore a formal neighborhood of a fixed point (d_V, ρ) of $d_{\mathcal{M}}$ becomes a formal pointed dg-manifold.

DEFINITION 6.2.4. The deformation theory of a dg-algebra V is controlled by this formal pointed dg-manifold.

Abusing notation we will denote the corresponding deformation functor by $Def_{\mathcal{M}}$ (more precisely we should use the formal neighborhood of a zero of $d_{\mathcal{M}}$ instead of $\mathcal{M}$). Since it will be always clear which zero of the vector field $d_{\mathcal{M}}$ is considered, such an abuse of notation should not lead to a confusion.

REMARK 6.2.5. Operads are algebras over the colored operad $\mathcal{OP}$. One can show that the deformation theories for an $\mathcal{OP}$-algebra $\widehat{P}$ described in the last two definitions are in fact equivalent.

Returning to our main conjecture, we can now prove a part of it.

THEOREM 6.2.6. *Let V be an algebra over an operad R. Then the "naive" deformation functor Def^V is isomorphic to $Def_{\mathcal{M}}$, where $\mathcal{M} = \mathcal{M}(P, V)$ is defined by means of any resolution $P \to R$.*

Proof. Let (C, m_C) be an Artin algebra. Then $Def_{\mathcal{M}}(C)$ is, by definition of $\mathcal{M}$, a set of isomorphims classes (modulo the action of $exp(\underline{Hom}(A, A) \otimes m_C)$) of a structure of P-algebra on $V \otimes C$, compatible with a structure of C-module, such that reduction modulo the maximal ideal m_C coincides with the structure induced via the quasi-isomorphism $P \to R$. Since $P = ... \to P^{-2} \to P^{-1} \to P^0 = R$ is a dg-operad with all P^{-n} placed in non-positive degrees, and since C has degree zero, we conclude that the action of the graded operad P is the same as the action of $P^0 = R$. Therefore, we obtain a natural morphism of functors $Def_{\mathcal{M}} \to Def^V$, which is an isomorphism by construction. ■

6.3. Example: A_∞-operad and A_∞-algebras. Let $V \in Vect_k^{\mathbf{Z}}$ and $m_n : V^{\otimes n} \to V[n-2], n \geq 2$ be a sequence of morphisms. It gives rise to an action on V of the free operad $P = Free_{\mathcal{OP}}(F)$ where

$$F(V) = \oplus_{n \geq 2} V^{\otimes n}[n-2].$$

Then $F(n) = k[S_n]m_n \otimes k[1]^{\otimes(n-2)}$. This notation means that we consider F_n as a space (with the grading shifted by $n-2$) of the regular representation of the group algebra of the symmetric group S_n. This space is generated by an element which we denote by m_n.

The differential $d_P \in g_P$ (equivalently, a structure of a dg-operad on P) is defined by the standard formulas:

$$d_P(m_2) = 0,$$

$$d_P(m_n)(v_1 \otimes ... \otimes v_n) = \sum_{k+l=n} \pm m_k(v_1 \otimes ... \otimes v_i \otimes m_l(v_{i+1} \otimes ... \otimes v_{i+l}) \otimes ... \otimes v_n), n > 2.$$

We do not specify signs in these formulas, since we do not need them here. Correct signs appear automatically from the interpretation of m_n as Taylor coefficients of the odd vector field on the non-commutative formal pointed manifold given in Chapter 6.

DEFINITION 6.3.1. The dg-operad $\mathcal{A}_\infty = (P, d_P)$ is called the A_∞-operad. Algebras over this dg-operad are called A_∞-algebras.

Deformations of an A_∞-algebra A are controlled by the truncated Hochschild complex

$$C_+^\bullet(A, A) = \prod_{n \geq 1} Hom_{Vect_k^{\mathbf{Z}}}(A^{\otimes n}, A)[-n]$$

More precisely, let A be a graded vector space. We defined (Chapter 4, Section 1.1) a graded vector space of Hochschild cochains of A as

$$C^\bullet(A, A) = \prod_{n \geq 0} Hom_{Vect_k^{\mathbf{Z}}}(A^{\otimes n}, A)[-n]$$

Then $C^\bullet(A, A)[1]$ can be equipped with the structure of a graded Lie algebra with the Gerstenhaber bracket (recall that the latter appears naturally if we interpret Hochschild cochains as derivations of the coalgebra $\oplus_{n \geq 0}(A[1])^{\otimes n}$).

Let us consider an element $m = (m_1, m_2 ...) \in C_+^\bullet(A, A)[1]$ of degree $+1$ such that $[m, m] = 0$. Such an element defines a differenitial $d = m_1$ on A, and the sequence $(m_2, m_3, ...)$ gives rise to a structure of an A_∞-algebra on (A, m_1).

Then we can make $C^\bullet(A, A)$ into a complex (Hochschild complex) with the differential $d_m = [m, \bullet]$. It was explained in Chapter 4 that in this way we get a differential-graded Lie algebra (DGLA for short) $(C^\bullet(A, A)[1], d_m)$. The truncated Hochschild complex $C_+^\bullet(A, A)[1]$ is a DGLA subalgebra. According to the general theory of Chapter 3 both DGLAs define formal pointed dg-manifolds, and therefore give rise to the deformation functors. This is a straightforward generalization of the deformation theory of associative algebras discussed in Chapter 4, Section 1.1.

EXERCISE 6.3.2. Prove that the DGLA $(C_+^\bullet(A, A)[1], d_m)$ controls the deformation theory of an A_∞-algebra (A, m).

Full Hochschild complex controls deformations of the A_∞-category with one object, such that its endomorphism space is equal to A. The deformation theory of A_∞-categories will be studied in the second volume of the book. Nevertheless we will refer to the formal dg-manifold associated with $C^\bullet(A, A)[1]$ as to the *moduli space of A_∞-categories*. Similarly, the formal dg-manifold associated with $C_+^\bullet(A, A)[1]$ will be called the *moduli space of A_∞-algebras*. (All the terminology assumes that we deform a given A_∞-algebra A).

The moduli space of A_∞-algebras is the same as $\mathcal{M}(\mathcal{A}_\infty, A)$ in the previous notation. Similarly we will denote the moduli space of A_∞-categories by

$\mathcal{M}_{cat}(\mathcal{A}_\infty, A)$. The natural inclusion of DGLAs $C^\bullet_+(A,A)[1] \to C^\bullet(A,A)[1]$ induces a dg-map $\mathcal{M}(\mathcal{A}_\infty, A) \to \mathcal{M}_{cat}(\mathcal{A}_\infty, A)$ (dg-map is a morphism of dg-manifolds).

Let us remark that the operad $\mathcal{A}_\infty$ gives rise to a free resolution of the operad As. Algebras over the latter are associative algebras without the unit.

Remark 6.3.3. It is interesting to describe deformation theories of free resolutions of the classical operads As, Lie, $Comm$. It seems that for an arbitrary free resolution P of either of these operads the following is true: $H^i(g_P) = 0$ for $i \neq 0$, $H^0(g_P) = k$. This one-dimensional vector space gives rise to the rescaling of operations, like $m_n \mapsto \lambda^n m_n$ in the case of A_∞-algebras.

6.4. Checking the conjecture for a free operad. Let G be a polynomial functor in the tensor category $\mathcal{C} = Vect_k$. We can think of G as of the functor from the groupoid of finite sets (all non-trivial morphisms are isomorphisms) to $\mathcal{C}$. Then we have a free operad $P = Free_{\mathcal{OP}}(G)$. For any finite set I with n elements we can write

$$P(I) = \oplus_{T \in \mathcal{T}(n)}(\prod_{v \in V_i(T)} G(N^{-1}(v)))/AutT.$$

Let A be a P-algebra. For simplicity we will assume that A is a trivial P-algebra, i.e. all maps $P(n) \otimes A^{\otimes n} \to A$ are trivial for $n \geq 2$. By definition we have a morphism $g : G(A) \to A$. Let $\overline{A} := G(A)[1] \oplus A$. We endow $\overline{A}$ with the differential $d := d_{\overline{A}}$ such that it is trivial on A and it is $id - \phi$ on $G(A)[1]$, where $\phi := g \circ [-1]$.

Proposition 6.4.1. *Natural morphism $f : P(\overline{A}) \to A$ such that $f(G(A)[1]) = 0$ and $f_{|A} = id_A$ defines a free resolution of the P-algebra A.*

Proof. Let $T \in \mathcal{T}(n)$ be a tree. We call *pre-tail* a vertex v such that $N^{-1}(v)$ consists of tail vertices only. We denote by $V_{prt}(T)$ the set of all pre-tail vertices. It is easy to see that if x_i is a generator of $P(\overline{A})$ then $f(d(x_i)) = 0$. What is left is to check that f is a quasi-isomorphism. In order to do that we rewrite $P(\overline{A})$ in a different way. Let us introduce a groupoid $\mathcal{T}(n)^{mod}$ of modified trees. The only difference with $\mathcal{T}(n)$ is that to each pre-tail vertex v we assign a number $\epsilon(v) \in \{0, 1\}$. Then

$$P(\overline{A}) = \oplus_{n \geq 0} \oplus_{T \in \mathcal{T}(n)^{mod}} W_T(A)[J_T],$$

where $W_T(A) = A^{\otimes V_t(T)} \otimes (\bigotimes_{v \in V_i(T)} G(N^{-1}(v)))$, and J_T is the set of pre-tail vertices v such that $\epsilon(v) = 1$.

Now the differential d can be written as $d = d_1 + d_2$ where d_1 changes the marking $\epsilon(v)$ from 1 to 0 at a pre-tail vertex v with $\epsilon(v) = 1$ (and then we take the sum over all such pre-tail vertices) and d_2 creates for any v as above a new vertex v_1 such that $N(v) = v_1$ with $-\phi$ inserted in v_1 (and then we again take the sum over all v). Notice that d_2 changes the number of internal vertices of T. Then we have an increasing filtration of $P(\overline{A})$ by subcomplexes $(F^m)_{m \geq 0}$ such that the corresponding graded components $gr_m(P(\overline{A})) = F^m/F^{m-1}$ are acyclic complexes (with the differential d_1) if $n > 0$. Indeed

$$gr_m(P(\overline{A})) = (\oplus_{T \in \mathcal{T}(n)^{mod}} W_T(A)) \otimes K^\bullet_T,$$

where $K_T^\bullet = \otimes_{v\in V_{prt}(T)}(k[1]\to k)$, and $k[1]\to k$ denotes the two-term complex with the differential, which is trivial on the summand k placed in degree zero and equal to $id\circ[-1]$ on the summand $k[1]$. It is easy to see that if $n>0$ then $K_T^\bullet$ is acyclic. On the other hand, for $n=0$ we obviously have the summand A. This proves that f is a quasi-isomorphism. ■

Let us compare now two approaches to the deformation theory of the P-algebra A. Since P is a free operad, we have a free resolution of it, which is $(P, d_P=0)$. The corresponding formal pointed dg-manifold in is the formal neighborhood of the point $(0,\phi)$ in the graded manifold $\mathcal{M} = \underline{End}(A)[1]\oplus \underline{Hom}(G(A),A)$. It is given by the DGLA $g_{\mathcal{M}} = \underline{End}(A)\oplus\underline{Hom}(G(A),A)[-1]$ (we have a Lie algebra $g_0=\underline{End}(A)$ and a g_0-module $\underline{Hom}(G(A),A)[-1]$, hence the graded Lie algebra structure on $g_{\mathcal{M}}$). We endow it with the trivial differential.

In the first approach we should consider the DGLA $\underline{Der}(P(\overline{A}))\simeq\underline{Hom}(G(A)[1]\oplus A, P(G(A)[1]\oplus A)$ (it is an isomorphism of graded vector spaces). Let us consider a graded Lie subalgebra $g_1\subset\underline{Der}(P(\overline{A}))$ which consists of derivations which preserve the set of generators $G(A)[1]\oplus A$ and such that they map $G(A)[1]$ into A. Clearly g_1 is a Lie subalgebra of $gl(G(A)[1]\oplus A)$ (it contains whole $\underline{End}(A)$ and linear maps $G(A)[1]\to A$). There is an obvious isomorphism of graded Lie algebras $g_{\mathcal{M}}\to g_1$. We leave to the reader to check that it is compatible with the differentials. Since g_1 is quasi-isomorphic to $\underline{Der}(P(\overline{A}))$, we conclude that the following result holds.

Proposition 6.4.2. *For algebras over a free operad two approaches to the deformation theory give isomorphic deformation functors.*

More precisely, we have checked that for *some* resolution of A and *some* resolution of P we obtained quasi-isomorphic formal pointed dg-manifolds. Independence of choices is a separate issue.

6.5. Homotopical actions of the Lie algebras of derivations. Let us recall the following construction (see Chapter 4, Section 2.3). Let g be a Lie algebra acting on a formal dg-manifold (Y,d_Y). This means that we have a homomorphism of Lie algebras $g\to Der(Y), \gamma\mapsto\hat{\gamma}$ where $Der(Y)$ is the Lie algebra of vector fields on Y preserving $\mathbf{Z}$-grading an d_Y.

We can make $Z=Y\times g[1]$ into a formal dg-manifold introducing an odd vector field by the following formula

$$d_Z(y,\gamma) = (d_Y(y)+\hat{\gamma}, [\gamma,\gamma]/2)$$

Then $[d_Z,d_Z]=0$. We can make $g[1]$ into a formal dg-manifold using the odd vector field $d_{g[1]}$ arising from the Lie bracket.

Exercise 6.5.1. The natural projection $(Z,d_Z)\to(g[1],d_{g[1]})$ is an epimorphism of formal dg-manifolds (i.e. a dg-bundle).

We see (cf. Chapter 4, Section 2.3) that we have a homotopical g-action on a formal dg-manifold (Y,d_Y), since we get a dg-bundle $\pi:(Z,d_Z)\to(g[1],d_{g[1]})$ together with an isomorphism of dg-manifolds $(\pi^{-1}(0),d_Z)\simeq(Y,d_Y)$.

Remark 6.5.2. It was pointed out in [Ko97-1] that in this case g acts on the cohomology of all complexes naturally associated with (Y,d_Y) (like the tangent space at a zero point of d_Y, the space of formal functions on Y, etc.).

Suppose that F is a polynomial functor in the category of **Z**-graded vector spaces, $P = Free(F), V \in Vect_k^{\mathbf{Z}}$. We apply the general scheme outlined above to the case $Y = \mathcal{M}(P, V), g = g_P$. Obviously g acts on the dg-manifold $\underline{Hom}(F(V), V)$, equipped with the trivial odd vector field.

Let us consider the graded vector space

$$\mathcal{N} = \underline{Hom}(V, V)[1] \oplus \underline{Hom}(F(V), V) \oplus g_P[1]$$

Let $d_V \in \underline{Hom}(V, V)[1]$ makes V into a complex, $\gamma = d_P \in g_P[1]$ satisfies the equation $[d_P, d_P] = 0$ and $\rho \in \underline{Hom}(F(V), V)$ makes V into a dg-algebra over (P, d_P).

We consider the formal neighborhood of the point (d_V, ρ, d_P) in $\mathcal{N}$, and define an odd vector field by the formula

$$d_{\mathcal{N}}(d_V, \rho, d_P) = (d_V^2, \xi(d_V, \rho) + \hat{d}_P, [d_P, d_P]/2)$$

The notation here is compatible with the one for $\mathcal{M}$.

One can check that $[d_{\mathcal{N}}, d_{\mathcal{N}}] = 0$. Thus the formal neighborhood becomes a formal dg-manifold. It controls deformations of pairs *(an operad, an algebra over this operad)*.

The natural projection $\pi : \mathcal{N} \to g_P[1]$ is a morphism of formal dg-manifolds . Here on $g_P[1]$ we use the odd vector field $d_{g_P[1]}$ defined by the Lie bracket. Then the formal scheme of zeros of $d_{g_P[1]}$ corresponds to the structures of a dg-operad on P. The fiber over a fixed point $x \in g_P[1]$ is a dg-manifold with the differential induced from $\mathcal{N}$. Then the formal neighborhood of a fixed point in $\pi^{-1}(x)$ controls deformations of $\widehat{P}$-algebras.

We conclude that the Lie algebra of derivations of an operad acts homotopically on the moduli space of algebras over this operad.

6.6. Independence of a choice of resolution of operad. It is natural to ask whether the deformation theory is independent of choices of resolutions. The situation here is quite different for the first and second approachs to the deformation theory of algebras over operads. The reason is that the quasi-isomorphism class of the DGLA $\underline{Der}(R(E))$ does not depend functorially of a choice of the cofibrant resolution $R(E) \to V$. This is not a big surprise: Lie algebra of vector fields on a manifold does not depend functorially of the manifold.

On the other hand, the formal pointed dg-manifold $\mathcal{M}(P, V)$ depends on P in a functorial way. More precisely, one has the following result.

PROPOSITION 6.6.1. *Let $P_1 \to P_2$ be a morphism of dg-operads. Then it induces a morphism of dg-manifolds $\mathcal{M}(P_1, V) \to \mathcal{M}(P_2, V)$.*

Proof. Exercise. ∎

THEOREM 6.6.2. *Let P_1 and P_2 be two cofibrant resolutions of an operad R, V be an R-algebra. Then $Def_{\mathcal{M}(P_1,V)} \simeq Def_{\mathcal{M}(P_2,V)}$.*

Proof. Let us recall that we have a morphism of dg-operads $\phi : P \to R$, where $P = Free_{\mathcal{OP}}(F)$ as a **Z**-graded operad, and ϕ is a quasi-isomorphism. Moreover:

a) F admits a filtration (as a polynomial functor) $F = \cup_{j \geq 1} F^{(j)}, F^{(j)} \subset F^{(j+1)}$ such that $d_P(F^{(0)}) = 0$ and $d_P(F^{(j)}) \subset Free_{\mathcal{OP}}(F^{(j-1)}), j \geq 1$;

b) $\phi : P \to R$ is a an epimorphism.

Since P_1 and P_2 are cofibrant resolutions of the same operad R, they are homotopy equivalent (as algebras over the colored operad $\mathcal{OP}$). More precisely, we notice that the homotopy theory of associative commutative dg-algebras developed in Chapter 3, Section 2.10, can be rephrased word-by-word to the case of cofibrant dg-algebras over an operad. By a cofibrant dg-algebra over an operad P we mean a dg-algebra, which is free graded P-algebra (i.e. if we forget the differentials). Then the main theorem of Chapter 3, Section 2.10 holds in this new setting, with the same proof. Moreover, the theory holds for algebras over colored operads, in particular, for algebras over the operad $\mathcal{OP}$. Thus we have homotopy equivalences $f : P_1 \to P_2$ and $g : P_2 \to P_1$. They induce morphisms of formal dg-manifolds $\mathcal{M}(P_1, V) \to \mathcal{M}(P_2, V)$ (see Chapter 3) and $\mathcal{M}(P_2, V) \to \mathcal{M}(P_1, V)$. Since f and g are homotopy equivalences, then the above morphisms of formal pointed dg-manifolds are mutually inverse on the cohomology. (WHY?) This concludes the proof. ■

7. Hochschild complex and Deligne's conjecture

There is a dg-operad of the type $\widehat{P}$ (i.e. it is free as a graded operad) acting naturally on the Hochschild complex of an arbitrary A_∞- algebra. Using this dg-operad we can develop the deformation theory of A_∞-algebras and relate it with the geometry of configurations of points in $\mathbf{R}^2$. This circle of questions is concentrated around so-called Deligne conjecture. There are several ways to formulate it as well as several independent proofs. An approach suggested in [KS2000] is motivated by the general philosophy of the previous section.

Namely, there is an operad M which acts naturally on the full Hochschild complex $C^\bullet(A, A)$ of an A_∞-algebra A as well as on $C_+^\bullet(A, A)$. There is a natural free resolution P of the operad M, so that $C := C^\bullet(A, A)$ becomes a $\widehat{P}$-algebra. Then we can say that there is a dg-map of the moduli space of A_∞-categories to the moduli space $\mathcal{M}(P, C)$ of structures of $\widehat{P}$-algebras on the graded vector space C.

From the point of view of deformation theory it is not very natural to make constructions of the type *algebraic structure* $\to$ *another algebraic structure* (like our construction $A_\infty-algebras \to M-algebras$). It is more natural to extend them to morphisms between the formal pointed dg-manifolds controlling the deformation theories of the structures. In fact there is an explicit dg-map $\mathcal{M}(\mathcal{A}_\infty, A) \to \mathcal{M}(P, C)$ as well as a dg-map $\mathcal{M}_{cat}(\mathcal{A}_\infty, A) \to \mathcal{M}(P, C)$, such that the one is obtained from another by the restriction from the moduli space of algebras to the moduli space of categories.

The operad $\mathcal{A}_\infty$ is augmented, i.e. equipped with a morphism of dg-operads $\eta :$ $\mathcal{A}_\infty \to Free_0$, where $Free_0$ is the trivial operad : $Free_0(1) = k, Free_0((n \neq 1)) = 0$. Since A (as any graded vector space) is an algebra over $Free_0$, it becomes also an algebra over $\mathcal{A}_\infty$. Any structure of an $\mathcal{A}_\infty$-algebra on A can be considered as a deformation of this trivial structure. Notice also that in the previous notation the augmentation morphism defines a point in the dg-manifold $\mathcal{M}(\mathcal{A}_\infty, C)$, where $C = C^\bullet(A, A)$. Therefore it is sufficient to work in the formal neighborhood of this point.

We can consider also the moduli space of structures of a complex on the graded vector space $C^\bullet(A, A)$, where A is an arbitrary graded vector space. It gives rise to a formal dg-manifold. There are natural morphisms to it from the formal dg-manifold of the moduli space of A_∞-categories and from the formal dg-manifold of the moduli space of structures of $\widehat{P}$-algebras on $C^\bullet(A,A)$. Theorem below combines all three morphisms discussed above into a commutative diagram. Let us make it more precise. First we formulate a simple general lemma, which will be applied in the case $V = C[2]$.

Let V be an arbitrary graded vector space, d_V an odd vector field on V (considered as a graded manifold) such that $[d_V, d_V] = 0$. Thus we get a dg-manifold. The graded vector space $H = H(V) = \underline{Hom}(V,V)[1]$ is a dg-manifold with $d_H(\gamma) = \gamma^2$. To every point $v \in V$ we assign a point in H by taking the first Taylor coefficient $d_V^{(1)}(v)$ of d_V at v. In this way we obtain a map $\nu : V \to H$.

LEMMA 7.0.3. *The map ν is a morphism of dg-manifolds.*

Proof. Let us write in local coordinates $x = (x_1, ..., x_n)$ the vector field $d_V = \sum_i \phi_i \partial_i$ where ∂_i denotes the partial derivative with respect to x_i, and ϕ_i are functions on V. Then the map ν assigns to a point x the matrix $M = (M_{ij}(x))$ with $M_{ij} = \partial_j \phi_i$. Then direct computation shows that the condition $[d_V, d_V] = 0$ implies that the vector field $\dot{x} = d_V(x)$ is mapped to the vector field $\dot{M} = d_H(M) = M^2$. ■

Let A be a graded vector space endowed with the trivial A_∞-structure, and $C = C^\bullet(A,A) = \prod_{n\geq 0} Hom_{Vect_k^{\mathbf{Z}}}(A^{\otimes n}, A)$ be the graded space of Hochschild cochains. Since $C[1]$ carries a structure of a graded Lie algebra (with the Gerstenhaber bracket), it gives rise to the structure of a dg-manifold on $C[2]$, which is the same as $\mathcal{M}_{cat}(\mathcal{A}_\infty, A)$. We will denote it by (X, d_X) (or simply by X for short).

Even for the trivial A_∞-algebra structure on A, we get a non-trivial P-algebra structure on C. The corresponding moduli space $\mathcal{M}(P,C)$ will be denoted by (Y, d_Y) (or Y for short).

There is a natural morphism of dg-manifolds $p : Y \to \underline{Hom}(C,C)[1] = H$ (projection of $Y = \mathcal{M}(P,C)$ to the first summand).

We omit the proof of the following theorem (see KoSo2000]).

THEOREM 7.0.4. *There exists a $GL(A)$-equivariant morphism of dg-manifolds $f : X \to Y$ such that $pf = \nu$.*

Moreover there is an explicit construction of the morphism.

Suppose that A is an A_∞-algebra. Geometrically the structure of an A_∞-algebra on the graded vector space A gives rise to a point $\gamma \in X = C[2], C = C^\bullet(A,A)$ such that $d_X(\gamma) = 0$. Indeed, the definition can be written as $[\gamma,\gamma] = 0$. Thus we get a differential in C (commutator with γ) making it into a complex. The structure of a complex on the graded vector space C gives rise to a zero of the field d_H in the dg-manifold $H = \underline{Hom}(C,C)[1]$. Theorem 1 implies that $f(\gamma)$ is a zero of the vector field $d_{\mathcal{M}(P,C)}$. Therefore the Hochschild complex $(C, [\gamma, \bullet])$ carries a structure of a dg-algebra over P.

7.1. Operad of little discs and Fulton-Macpherson operad. We fix an integer $d \geq 1$. Let us denote by G_d the $(d+1)$-dimensional Lie group acting on $\mathbf{R}^d$ by affine transformations $u \mapsto \lambda u + v$, where $\lambda > 0$ is a real number and $v \in \mathbf{R}^d$ is a vector. This group acts simply transitively on the space of closed discs in $\mathbf{R}^d$ (in the usual Euclidean metric). The disc with center v and with radius λ is obtained from the standard disc

$$D_0 := \{(x_1,\ldots,x_d)\in \mathbf{R}^d |\, x_1^2+\cdots+x_d^2\le 1\}$$

by a transformation from G_d with parameters (λ, v).

Definition 7.1.1. The little discs operad $E_d = \{E_d(n)\}_{n\ge 0}$ is a topological operad defined such as follows:
1) $E_d(0) = \emptyset$,
2) $E_d(1) =$ point $= \{\mathrm{id_{E_d}}\}$,
3) for $n \ge 2$ the space $E_d(n)$ is the space of configurations of n disjoint discs $(D_i)_{1\le i\le n}$ inside the standard disc D_0.

The composition $E_d(k)\times E_d(n_1)\times\cdots\times E_d(n_k)\to E_d(n_1+\cdots+n_k)$ is obtained by applying elements from G_d associated with discs $(D_i)_{1\le i\le k}$ in the configuration in $E_d(k)$ to configurations in all $E_d(n_i)$, $i = 1,\ldots,k$ and putting the resulting configurations together. The action of the symmetric group S_n on $E_d(n)$ is given by renumeration of indices of discs $(D_i)_{1\le i\le n}$.

The space $E_d(n)$ is homotopy equivalent to the configuration space of n pairwise distinct points in $\mathbf{R}^d$.

There is an obvious continuous map $E_d(n)\to \mathsf{Conf_n}(Int(D_0))$ which associates to a collection of disjoint discs the collection of their centers. This map induces a homotopy equivalence because its fibers are contractible.

The little discs operad and homotopy equivalent little cubes operad were introduced in topology by J. P. May in order to describe homotopy types of iterated loop spaces.

The Fulton-Macpherson operad defined below is homotopy equivalent to the little discs operad.

For $n \ge 2$ we denote by $\tilde{E}_d(n)$ the quotient space of the configuration space of n points in $\mathbf{R}^d$

$$\mathsf{Conf_n}(\mathbf{R}^d) := \{(x_1,\ldots,x_n)\in(\mathbf{R}^d)^n |\, x_i\ne x_j \text{ for any } i\ne j\}$$

by the action of the group $G_d = \{x\mapsto \lambda x+v |\, \lambda\in\mathbf{R}_{>0},\ v\in\mathbf{R}^d\}$. The space $\tilde{E}_d(n)$ is a smooth manifold of dimension $d(n-1)-1$. For $n = 2$, the space $\tilde{E}_d(n)$ coincides with the $(d-1)$-dimensional sphere S^{d-1}. There is an obvious free action of S_n on $\tilde{E}_d(n)$. We define the spaces $\tilde{E}_d(0)$ and $\tilde{E}_d(1)$ to be empty. The collection of spaces $\tilde{E}_d(n)$ does not form an operad because there is no identity element, and compositions are not defined.

Now we are ready to define the operad $FM_d = \{FM_d(n)\}_{n\ge 0}$

The components of the operad FM_d are

1) $FM_d(0) := \emptyset$,

2) $FM_d(1) =$point,

3) $FM_d(2) = \tilde{E}_d(2) = S^{d-1}$,

4) for $n\ge 3$ the space $FM_d(n)$ is a manifold with corners, its interior is $\tilde{E}_d(n)$, and all boundary strata are certain products of copies of $\tilde{E}_d(n')$ for $n' < n$.

The spaces $FM_d(n), n\ge 2$ can be defined explicitly.

Definition 7.1.2. For $n\ge 2$, the manifold with corners $FM_d(n)$ is the closure of the image of $\tilde{E}_d(n)$ in the compact manifold $\left(S^{d-1}\right)^{n(n-1)/2}\times[0,+\infty]$ under the map

$G_d \cdot (x_1, \ldots, x_n) \mapsto (\left(\frac{x_j - x_i}{|x_j - x_i|} \right)_{1 \le i < j \le n}, \frac{|x_i - x_j|}{|x_i - x_k|})$
where i, j, k are pairwise distinct indices.

One can define the natural structure of operad on the collection of spaces $FM_d(n)$. We skip here the obvious definition.

It is easy to check that in this way we obtain a topological operad (in fact an operad in the category of real compact piecewise algebraic sets defined in Appendix). We call it the *Fulton-Macpherson operad* and denote by FM_d.

Set-theoretically, the operad FM_d is the same as the free operad generated by the collection of sets $(\tilde{E}_d(n))_{n \ge 0}$ endowed with the S_n-actions as above.

Let us make few remarks about geometric origin of the operad P, which is a free resolution of the operad M (see previous section). It is related to the configuration space of discs inside of the unit disc in the plane. More precisely, the operad $Chains(E_2)$ of singular chains on the little discs operad is quasi-isomorphic to $\widehat{P}$. In fact there is a morphism $\widehat{P} \to Chains(E_2)$ which gives the homotopy equivalence (to be more precise it is easier to construct it for the operad $Chains(FM_2)$ which is quasi-isomorphic to E_2). Then using the fact that both dg-operads are free as graded operads, one can invert this quasi-isomorphism. This gives a structure of an $Chains(E_2)$-algebra on the Hochschild complex of an A_∞-algebra. This result is known as Deligne's conjecture.

More precisely, it is stated such as follows.

Conjecture 7.1.3. There exists a natural action of the operad $Chains(C_2)$ on the Hochschild complex $C^*(A, A)$ for an arbitrary associative algebra A.

There are several proofs of this conjecture (see for ex. [KoSo2000]). Having in mind this result we can give the following definition.

Definition 7.1.4. A graded vector space V is called a d-algebra if it is an algebra over the operad $Chains(E_d)$ (we take singular chains of the topological operad of little d-dimensional discs).

Then Deligne's conjecture says that Hochschild complex of an associative algebra (more general, A_∞-algebra) is a 2-algebra.

The moduli space $\mathcal{M}(P, C)$ can be thought of as a moduli space of structures of a 2-algebra on a graded vector space C. Then the theorem above says that there is a $GL(A)$-equivariant morphism of the moduli space of A_∞-categories with one object to the moduli space of 2-algebras.

Let $g_P = \underline{Der} P$ be as before the DGLA of derivations of $\widehat{P}$. Then g_P acts on the moduli space of $\widehat{P}$-algebras.

Remark 7.1.5. There is a natural action of the so-called Grothendieck-Tiechmüller group on the rational homotopy type of the Fulton-Macpherson operad for $\mathbf{R}^2$. Now we remark that our theorem gives rise to a morphism of L_∞-algebras $Lie(GT)[1] \to (g_P, [d_P, \bullet])$ where GT is the Grothendieck-Teichmüller group.

Therefore one has a homotopical action of the Lie algebra $Lie(GT)$ on the moduli space of $\widehat{P}$-algebras.

7.2. Higher-dimensional generalization of Deligne's conjecture. For $d > 0$ the notion of d-algebra was introduced by Getzler and Jones. By definition, a 0-algebra is just a complex. An A_∞-version of Deligne's conjecture says that this Hochschild complex carries naturally a structure of 2-algebra, extending the structure of differential graded Lie algebra. It has a baby version in dimension $(0+1)$: if A is a vector space (i.e. 0-algebra concentrated in degree 0) then the Lie algebra of the group of affine transformations

$$Lie(Aff(A)) = End(A) \oplus A$$

has also a natural structure of an associative algebra, in particular it is a 1-algebra. The product in $End(A) \oplus A$ is given by the formula

$$(\phi_1, a_1) \times (\phi_2, a_2) := (\phi_1\phi_2, \phi_1(a_2)) \ .$$

The space $End(A) \oplus A$ plays the rôle of the Hochschild complex in the case $d = 0$.

Now we introduce the notion of *action* of a $(d+1)$-algebra on a d-algebra. It is convenient to formulate it using colored operads. Namely, there is a colored operad with two colors such that algebras over this operad are pairs (g, A) where g is a Lie algebra and A is an associative algebra on which g acts by derivations.

Let us fix a dimension $d \geq 0$. Denote by $\sigma : \mathbf{R^{d+1}} \to \mathbf{R^{d+1}}$ the reflection

$$(x_1, \ldots, x_{d+1}) \mapsto (x_1, \ldots, x_d, -x_{d+1})$$

at the coordinate hyperplane, and by H_+ the upper-half space

$$\{(x_1, \ldots, x_{d+1}) |\, x_{d+1} > 0\}$$

Definition 7.2.1. For any pair of non-negative integers (n, m) we define a topological space $SC_d(n, m)$ as
1) the empty space $\emptyset$ if $n = m = 0$,
2) the one-point space if $n = 0$ and $m = 1$,
3) in the case $n \geq 1$ or $m \geq 2$, the space of configurations of $m + 2n$ disjoint discs $(D_1, \ldots, D_{m+2n})$ inside the standard disc $D_0 \subset \mathbf{R^{d+1}}$ such that $\sigma(D_i) = D_i$ for $i \leq m$, $\sigma(D_i) = D_{i+n}$ for $m+1 \leq i \leq m+n$ and such that all discs $D_{m+1}, \ldots, D_{m+n}$ are in the upper half space H_+.

The reader should think about points of $SC_d(n, m)$ as about configurations of m disjoint semidiscs $(D_1 \cap H_+, \ldots, D_m \cap H_+)$ and of n discs $(D_{m+1}, \ldots, D_{m+n})$ in the standard semidisc $D_0 \cap H_+$. The letters "SC" stand for "Swiss Cheese" [V]. Notice that the spaces $SC_d(0, m)$ are naturally isomorphic to $C_d(m)$ for all m. One can define composition maps analogously to the case of the operad C_d:

$$SC_d(n, m) \times \big(C_{d+1}(k_1) \times \cdots \times C_{d+1}(k_n)\big) \times \big(SC_d(a_1, b_1) \times \cdots \times SC_d(a_m, b_m)\big)$$
$$\to SC_d(k_1 + \cdots + k_n + a_1 + \cdots + a_m, b_1 + \cdots + b_m)$$

Definition 7.2.2. The colored operad SC_d has two colors and consists of collections of spaces

$$\big(SC_d(n, m)\big)_{n,m \geq 0}, \quad \big(C_{d+1}(n)\big)_{n \geq 0},$$

and appropriate actions of symmetric groups, identity elements, and of all composition maps.

As before, we can pass from a colored operad of topological spaces to a colored operad of complexes using the functor $Chains$.

Definition 7.2.3. An action of a $(d+1)$-algebra B on a d-algebra A is, on the pair (B, A), a structure of algebra over the colored operad $Chains(CS_d)$, compatible with the structures of algebras on A and on B.

The *generalized Deligne conjecture* says that for every d-algebra A there exists a universal (in an appropriate sense) $(d+1)$-algebra acting on A.

A version of this conjecture was proved in [HKV03].

7.3. Digression: homotopy theory and deformation theory. Let P be an operad of complexes, and $f : A \to B$ be a morphism of two P-algebras.

DEFINITION 7.3.1. we say that f is a quasi-isomorphism if it induces an isomorphism of the cohomology groups of A and B considered just as complexes.

Two algebras A and B are called *homotopy equivalent* iff there exists a chain of quasi-isomorphisms

$$A = A_1 \to A_2 \leftarrow A_3 \to \cdots \leftarrow A_{2k+1} = B$$

One can define a new structure of category on the collection of P-algebras in which quasi-isomorphic algebras become equivalent. There are several ways to do it, using either Quillen's machinery of homotopical algebra (see [Q]), or using a free resolution of the operad P, or some simplicial constructions. We are going to discuss the details in the second volume of the book. In the case of differential graded Lie algebras, morphisms in the homotopy category are L_∞-morphisms modulo a homotopy between morphisms. More generally, morphisms in the homotopy category of P-algebras are connected components of certain topological spaces, exactly as in the usual framework of homotopy theory.

In the case when the operad P satisfies some mild technical conditions, one can transfer the structure of a P-algebra by quasi-isomorphisms of complexes. In particular, one can make the construction described in the following lemma.

LEMMA 7.3.2. *Let P be an operad of complexes, such that if we consider P as an operad just of $\mathbf{Z}$-graded vector spaces, it is free and generated by operations in ≥ 2 arguments. Let A be an algebra over P, and let us choose a splitting of A considered as a $\mathbf{Z}$-graded space into the direct sum*

$$A = H^*(A) \oplus V \oplus V[-1], \ \ (V[-1])^k := V^{k-1}$$

endowed with a differential $d(a \oplus b \oplus c) = 0 \oplus 0 \oplus b[-1]$. Then there is a canonical structure of a P-algebra on the cohomology space $H^\bullet(A)$, and this algebra is homotopy equivalent to A.

Notice that the operad $\mathbf{Q} \otimes Chains(C_d)$ is free as an operad of $\mathbf{Z}$-graded vector spaces over $\mathbf{Q}$. This is evident because the action of S_n on $C_d(n)$ is free and the composition of morphisms in C_d are embeddings.

We associated with any operad P of complexes and with any P-algebra A a differential graded Lie algebra (or more generally, a L_∞-algebra $Def(A)$ (more precisely, we constructed a formal pointed dg-manifold). This DGLA is defined canonically up to a quasi-isomorphism (equivalently, up to homotopy). It controls the deformations of P-algebra structure on A. As we have already explained, there are several equivalent constructions of $Def(A)$ using either resolutions of A or resolutions of the operad P. Morally, $Def(A)$ is the Lie algebra of derivations in homotopy sense of A. For example, if P is an operad with zero differential then $Def(A)$ is quasi-isomorphic to the differential graded Lie algebra of derivations of $\tilde{A}$ where $\tilde{A}$ is any free resolution of A.

Differential graded Lie algebras $Def(A_1)$ and $Def(A_2)$ are quasi-isomorphic for homotopy equivalent P-algebras A_1 and A_2.

7.4. Universal Hochschild complex and deformation theory. If A a d-algebra then the shifted complex $A[d-1]$,

$$(A[d-1])^k := A^{(d-1)+k}$$

carries a natural structure of L_∞-algebra. It comes from a homomorphism of operads in homotopy sense from the twisted by $[d-1]$ operad $Chains(C_d)$ to the operad Lie. In order to construct such a homomorphism one can use fundamental chains of all components of the Fulton-MacPherson operad.

Moreover, $A[d-1]$ maps as L_∞-algebra to $Def(A)$, i.e. $A[d-1]$ maps to "inner derivations" of A. These inner derivations form a Lie ideal in $Def(A)$ in homotopy sense.

Conjecture 7.4.1. The quotient homotopy Lie algebra $Def(A)/A[d-1]$ is naturally quasi-isomorphic to $Hoch(A)[d]$.

In the case when $d=0$ and the complex A is concentrated in degree 0, the Lie algebra $Def(A)$ is $End(A)$, i.e. it is the Lie algebra of linear transformations in the vector space A.

Lemma 7.4.2. *The Hochschild complex of* 0*-algebra* A *is* $A\oplus \mathsf{End(A)}$ *(placed in degree* 0*).*

Proof. First of all, the colored operad SC_0 is quasi-isomorphic to its zero-homology operad $H_0(SC_0)$ because all connected components of spaces $(SC_0(n,m))_{n,m\geq 0}$ and of $(C_1(n))_{n\geq 0}$ are contractible. By general reasons this implies that we can replace SC_0 by $H_0(SC_0)$ in the definition of the Hochschild complex given above. The H_0-version of a 1-algebra is an associative non-unital algebra, and the H_0-version of an action is the following:

1) an action of an associative non-unital algebra B on vector space A (it comes from the generator of $\mathbf{Z}=\mathbf{H_0}(\mathbf{SC_0}(\mathbf{1},\mathbf{1}))$,

2) a homomorphism from B to A of B-modules (coming from the generator of $\mathbf{Z}=\mathbf{H_0}(\mathbf{SC_0}(\mathbf{1},\mathbf{0}))$.

It is easy to see that to define an action as above is the same as to define a homomorphism of non-unital associative algebras from B to $End(A)\oplus A$. Thus, the Hochschild complex is (up to homotopy) isomorphic to $End(A)\oplus A$.

Let us continue the explanation for the case $d=0$. The L_∞-algebra $Hoch(A)$ is quasi-isomorphic to the Lie algebra of affine transformations on A. The homomorphism $Def(A)\to Hoch(A)$ is a *monomorphism*, but in homotopy category every morphism of Lie algebras can be replaced by an epimorphism. The abelian graded Lie algebra $A[-1]$ is the "kernel" of this morphism. More precisely, the Lie algebra $Def(A)=(A)$ is quasi-isomorphic to the following differential graded Lie algebra g: as a $\mathbf{Z}$-graded vector space it is

$$End(A)\oplus A\oplus A[-1].$$

In other words the graded components of g are $g^0=End(A)\oplus A$, $g^1=A$, $g^{\neq 0,1}=0$. The nontrivial components of the Lie bracket on g are the usual bracket on $End(A)$ and the action of $End(A)$ on A and on $A[-1]$. The only nontrivial component of the differential on g is the shifted by $[1]$ identity map from A to $A[-1]$. The evident homomorphism

$$g\to End(A)$$

is a homomorphism of differential graded Lie algebras, and also a quasi-isomorphism. There is a short exact sequence of dg-Lie algebras

$$0 \to A[-1] \to g \to End(A) \oplus A \to 0$$

This concludes the proof. ■

In the case $d = 1$ the situation is similar. The deformation complex of an associative algebra A is the following *subcomplex* of the shifted by [1] Hochschild complex:

$$Def(A)^n := Hom_{Vect_k}(A^{\otimes(n+1)}, A) \text{ for } n \geq 0; \quad Def^{<0}(A) := 0$$

The deformation complex is quasi-isomorphic to the L_∞-algebra g which as $\mathbf{Z}$-graded vector space is

$$Def(A) \oplus A \oplus A[1].$$

The Hochschild complex of A is a *quotient* complex of g by the homotopy Lie ideal A.

7.5. Formality of the operad of little discs. Analogously to the case of algebras, we can speak about quasi-isomorphisms of operads in the category of complexes of vector spaces. Indeed, operads are just algebras over the colored operad $\mathcal{OP}$.

DEFINITION 7.5.1. A morphism $f : P_1 \to P_2$ between two dg-operads is called a quasi-isomorphism if the morphisms of complexes $f(n) : P_1(n) \to P_2(n)$ induce isomorphisms of cohomology groups for all n.

Conjecturally, homotopy categories and deformation theories of algebras over quasi-isomorphic operads are equivalent.

EXAMPLE 7.5.2. a) The operad Lie is quasi-isomorphic to the operad L_∞.

b) The operad As is quasi-isomorphic to the operad $Chains(C_1)$, and also to the operad $\mathcal{A}_\infty$.

In this subsection we are going to discuss the following important result.

THEOREM 7.5.3. *The operad $Chains(C_d) \otimes \mathbf{R}$ of complexes of real vector spaces is quasi-isomorphic to its cohomology operad endowed with zero differential.*

In general, differential graded algebras which are quasi-isomorphic to their cohomology endowed with zero differential, are called formal. Classical example is the de Rham complex of a compact Kähler manifold. The result of Deligne-Griffiths-Morgan-Sullivan (see [DGMS 75]) says that this algebra is formal as differential graded commutative associative algebra. The above theorem says that $Chains(C_d) \otimes \mathbf{R}$ is formal as an algebra over the colored operad $\mathcal{OP}$. We are not going to prove the formality theorem here, referring the reader to [Ko99] and [T98].

8. Deformation theory of algebras over PROPs

First, we would like to illustrate how the language of colored operads can be used in order to describe PROPs.

Let $Vect_k$ be the category of k-vector spaces, considered as a symmetric monoidal category. Then for a finite set I we have the category $Vect^I$ consisting of families $(V_i)_{i \in I}$ of k-vector spaces. We have the notion of a polynomial functor $F : Vect^I \to Vect^I$. It is given by a "Taylor series in many variables with coefficients which are representations of symmetric groups" (see the corresponding definition for operads). Polynomial functors form a monoidal category, and colored operads are monoids in this category. Every such a monoid defines a triple in the category $Vect^I$. Then one can speak about algebras over a colored operad. In this way a k-linear PROP (see Chapter 2) becomes an algebra over the colored operad $\mathcal{PR} = (\mathcal{PR}_{(m_{k,k'}),n,n'})$. In order to describe the components of this colored operad we will use the language of graphs.

First of all, we have now sequences $(U_{n,m})_{n,m\geq 0}$ of $S_n \times S_m$-modules.

This means that instead of the category $Vect^{I_0}$ we have a category $Vect^{I_0\times I_0}$ of sequences of vector spaces parametrized by pairs of Young diagrams.

Instead of the groups $S_{(m_k),n}$ which appear in the definition of a colored operad we now have groups

$$S_{m_{k,k'},n,n'} = \prod_{k\geq 0}\prod_{k'\geq 0} S_n \times S_{n'} \times (S_{m_{k,k'}} \ltimes (S_k \times S_{k'})^{m_{k,k'}})$$

given for each sequence of non-negative integers $m_{k,k'}$ such that
$\sum_{k,k'\geq 0} m_{k,k'} < \infty$

The component $\mathcal{PR}_{(m_{k,k'}),n,n'}$ is a k-vector space generated by the classes of isomorphisms of oriented graphs with input vertices numbered from 1 to n, with output vertices numbered from 1 to n', with $m_{k,k'}$ internal vertices numbered from 1 to $m_{k,k'}$ such that every such a vertex has k input edges and k' output edges. We also require that for every internal vertex v all edges incoming to v are numbered and all edges outcoming from v are numbered.

Then the groups $S_{m_{k,k'},n,n'}$ act on the graphs in a way similar to the case of $\mathcal{OP}$ described before. Composition maps are given by the procedure of inserting of a graph into an internal vertex. Again it is similar to the case of the colored operad $\mathcal{OP}$. Clearly, algebras over $\mathcal{PR}$ are k-linear PROPs.

Let us now return to the deformation theory. Let H be a k-linear PROP and V be an H-algebra. How to describe the deformation theory of V? If H was an operad or colored operad we would have three approaches to the deformation theory of V: the "naive one" (which is basically, just a statement of the problem), the one via free resolution of V and the one via free resolution of H. In the case of PROPs we have the naive one, and the one via resolution of PROPs. Indeed, the notion of a free algebra over a PROP does not exist. For example, there is no obvious way to define free Hopf algebras. Therefore, in order to construct a formal pointed dg-manifold controlling the deformation theory of V, one needs to construct a dg-PROP P_H which is free as a graded PROP, as well as a surjective quasi-isomorphism $P_H \to H$ of dg-PROPs (we endow H with zero differential). Then we can construct a formal pointed dg-manifold $\mathcal{M}(P_H, V)$ similarly to the one constructed previously for a resolution of a k-linear operad. Since we want the deformation theory to be independent of P_H, we want P_H to be a cofibrant resolution (i.e. it should be a cofibrant algebra over the colored operad $\mathcal{PR}$).

8.0.1. *PROP of bialgebras.* In order to make this approach practical one needs to construct cofibrant resolutions of PROPs. Of course, one can use Boardman-Vogt approach, similarly to the case of operads. But in this way we obtain resolutions which are too big. Unfortunately, very few resolutions of PROPs are known. Here we can mention the case when H is a PROP of Hopf algebras (more precisely, bialgebras, since the existence of an antipode is not required). It was studied in

[M02], [MV03], where the resolution of the PROP $\mathcal{B}$ of bialgebras was suggested. Main result can be stated such as follows.

THEOREM 8.0.4. *There exist a cofibrant resolution $P_{\mathcal{B}} \to \mathcal{B}$ of the PROP of bialgebras, with generators $\gamma_{m,n}$, where $m, n \geq 1, n+m \geq 3$ such that $\gamma_{1,1}$ corresponds to the product in a bialgebra and $\gamma_{1,2}$ corresponds to the coproduct.*

Unfortunately, there is no explicit formula for the action of the differential on the generators.

The above theorem says that $\mathcal{B}(n,m) = k^{\mathcal{SB}(n,m)}$, where $\mathcal{SB}(n,m)$ is a PROP in the symmetric monoidal category *Sets*. This set-theoretical PROP has generators parametrized by graphs with n numbered inputs and m numbered outputs.

CONJECTURE 8.0.5. There is a PROP $\mathcal{CWB}$ in the symmetric monoidal category of CW-complexes, which is free as a PROP in the category of sets and generated by cells $D_{n,m}$ of dimension $n+m-3$. There is a quasi-isomorphism of dg-PROPs $f : Chains(\mathcal{CWB}) \simeq P_{\mathcal{B}}$ such that $f(D_{n,m}) = \gamma_{n,m}$.

One can define a k-linear PROP $\frac{1}{2}\mathcal{B}$, called the PROP of 1/2-*bialgebras*. Algebras over this PROP are called 1/2-bialgebras. The idea was to kill the bialgebra relation $\Delta(ab) = \Delta(a)\Delta(b)$, which is not quadratic. For the PROP $\mathcal{B}$ this means to imposing the condition $\gamma_{2,2} = 0$, where $\gamma_{2,2}$ corresponds to the graph with two inputs, two outputs and two internal vertices, which are joined by a single edge (thus each of these two internal vertex has the total valency equal to 3). Then the PROP $\frac{1}{2}\mathcal{B}$ has a cofibrant resolution $P_{\frac{1}{2}\mathcal{B}}$ generated by the elements $\delta_{n,m}, n, m \geq 1, n+m \geq 3$ of degree $3-(n+m)$, which are parametrized by graphs with n inputs, m-outputs and have the only vertex. Differential d acts on $\delta_{n,m}$ by inserting an internal edge. We do not need an explicit formula.

Notice that the PROP $\mathcal{B}$ is a flat deformation of the PROP $\frac{1}{2}\mathcal{B}$. In order to see this, we introduce a family $\mathcal{B}_h$ of PROPs over $k[h]$, such that for $h = 0$ we get $\frac{1}{2}\mathcal{B}$ and for $h = 1$ we get $\mathcal{B}$. Algebras over $\mathcal{B}_h$ are vector spaces V equipped with the product $m : V \otimes V \to V$, coproduct $\Delta : V \to V \otimes V$, such that m is associative, Δ is coassociative and the compatibility relation is

$$\Delta \circ m - h(m \otimes m) \circ \sigma_{23} \circ (\Delta \otimes \Delta) = 0.$$

Here $\sigma_{23} : V^{\otimes 4} \to V^{\otimes 4}$ is the linear map such that $\sigma_{23}(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = v_1 \otimes v_3 \otimes v_2 \otimes v_4$.

The following result of [M02] shows all this can be applied to the deformation theory.

THEOREM 8.0.6. *There is a cofibrant resolution $P_{\mathcal{B}} \to \mathcal{B}$ of the PROP of bialgebras such that $P_{\mathcal{B}}(n,m) \simeq P_{\frac{1}{2}\mathcal{B}}(n,m)$ as graded vector spaces, and $d_{P_{\mathcal{B}}} = d_{P_{\frac{1}{2}\mathcal{B}}} + \sum_{l \geq 1} d_l$.*

In other words, there is a cofibrant resolution of the PROP of bialgebras which is a flat deformation of a cofibrant resolution of the PROP of 1/2-bialgebras. Then the formal pointed dg-manifold controlling the deformation theory of a bialgebra V should be a flat deformation of the formal pointed dg-manifold controlling the deformation theory of some 1/2-bialgebra V_0.

8.0.2. *PROP of Lie bialgebras.* Let g be a k-vector space, which we will assume finite-dimensional for simplicity.

DEFINITION 8.0.7. A Lie bialgebra structure on g is given by a pair of linear maps $b : g \wedge g \to g, b(x, y) := [x, y]$ called a Lie bracket and $\varphi : g \to g \wedge g$ called a Lie cobracket such that

1) b makes g into a Lie algebra;
2) φ makes the dual vector space g^* into a Lie algebra;
3) φ is a 1-cocycle, i.e. $\varphi([x, y]) = ad_x(\varphi(y)) - ad_y(\varphi(x))$.

Lie bialgebras were introduced by Drinfeld in the early 80's. They play an important role in the theory of quantum groups. The latter is beyond the scope of this book (we refer the reader to [KSo98]). Here we describe the k-linear PROP $\mathcal{LB}$ such that $\mathcal{LB}$-algebras are Lie bialgebras.

We set $\mathcal{LB}(m, 0) = \mathcal{LB}(0, n) = 0$. Suppose that $m, n \geq 1$. Let us consider a k-vector space $G(m, n)$ spanned by the isomorphism classes of finite oriented directed trivalent graphs with m inputs numbered from 1 to m and n outputs numbered from 1 to n. The word *direction* means that a direction is chosen for each edge, so that inputs are directed inward and outputs are directed outward. An *orientation* is an extra datum, which is a choice of a sign $+1$ or -1 for each edge. In particular, for a graph Γ we have an opposite graph Γ^{op} with an opposite orientation. In order to define a k-vector space $\mathcal{LB}(m, n)$ we first factorize $G(m, n)$ by the relation $\Gamma + \Gamma^{op} = 0$. Then we impose three extra relations corresponding to the conditions 1)-3) above. In order to to that one depicts the Lie bracket and cobracket by the same graphs used for the product and coproduct of a bialgebra. We leave this as an exercise to the reader. Then $\mathcal{LB}(m, n)$ is a vector space which is the quotient of $G(m, n)$ by the above relations. It follows from the definition that if g is an algebra over the PROP $\mathcal{LB}$ then g carries a structure of Lie bialgebra.

Now one can repeat for the PROP of Lie bialgebras all what we did for the PROP of bialgebras. In particular, one can introduce the notion of 1/2 Lie bialgebras, which are algebras over the corresponding PROP. Resolution of the PROP $\mathcal{LB}$ is a formal deformation of the resolution of the PROP of 1/2 Lie bialgebras. They are described in [MV03].

CHAPTER 6

A_∞-algebras and non-commutative geometry

1. Motivations

In this section we motivate the transition from k-linear categories to A_∞-categories. We remind to the reader that A_∞-categories will be discussed in detail in the second volume of the book. Nevertheless we think that one needs such a motivation even if one wants to work with A_∞-algebras only. In fact an A_∞-algebra is the same as A_∞-category with one object. Many ideas and constructions of this chapter admit straightforward generalizations to A_∞-categories.

1.1. From associative algebras to abelian categories. Let $\mathbf{k}$ be a commutative ring with the unit. Then one can defines the category of associative unital algebras over $\mathbf{k}$. Geometrically associative algebras correspond to "affine non-commutative spaces". One can generalize associative algebras, considering $\mathbf{k}$-linear categories. In a sense, $\mathbf{k}$-linear categories are "algebras with many objects". Indeed, a $\mathbf{k}$-linear category with one object is an associative algebra. On the other hand, a $\mathbf{k}$-linear category $\mathcal{C}$ with finitely many objects is the same as an associative $\mathbf{k}$-algebra with the finite set of commuting idempotents. Indeed, let $Ob(\mathcal{C}) = I$. We define an algebra $A = \oplus_{i,j\in I} Hom(i,j)$. Then A is a $\mathbf{k}$-algebra with the unit $1_A = \oplus_{i\in I} id_i$ and the multiplication given by the composition of morphisms. The elements $\pi_i = id_i \in A$ are commutting idempotents: $\pi_i^2 = \pi, \pi_i\pi_j = \pi_j\pi_i, \oplus_{i\in I}\pi_i = 1_A$. This construction gives rise to a homomorphism $\mathbf{k}^I \to A$ of algebras with the unit.

Conversely, suppose we are given an associative unital $\mathbf{k}$-algebra A, and $(\pi_i)_{i\in I}$ is a finite set of commuting idempotents in A, such that $\oplus_{i\in I}\pi_i = 1_A$. Then we can reconstruct a $\mathbf{k}$-linear category $\mathcal{C}$. To do this we set $Ob(\mathcal{C}) = I$, $Hom_{\mathcal{C}}(i,j) = \pi_i A \pi_j$. The composition of morphisms is defined in the obvious way.

Thus we see that there are "very small" linear categories which generalize associative algebras. Next step is to consider $\mathbf{k}$-linear *additive* categories. Now one has finite direct sums $\oplus_{i\in I} V_i$, including the case $I = \emptyset$. More precisely, an additive category admits finite sums and finite products and they coincide.

EXAMPLE 1.1.1. The category $A-mod$ of left modules over an associative ring, or some subcategories of the latter (like the category of free modules). Another example is given by the category of vector bundles over a given smooth manifold.

If we are given a $\mathbf{k}$-linear category $\mathcal{C}$, we can construct its additive envelope, so that the resulting category $\mathcal{C}^{(1)}$ is an additive category (i.e. admits finite direct sums). Namely, we define an object of $\mathcal{C}^{(1)}$ to be a finite family $(X_i)_{i\in I}$ of objects of $\mathcal{C}$. We define $Hom_{\mathcal{C}^{(1)}}((X_i),(Y_j)) = \oplus_{i,j} Hom_{\mathcal{C}}(X_i, Y_j)$. Composition of morphisms is defined by the matrix product and compositions in $\mathcal{C}$.

Next step is to consider *abelian* categories.

DEFINITION 1.1.2. Abelian category is an additive category which admits finite limits and colimits (equivalently, every morphism has a kernel and cokernel) and the coimage of any morphism is isomorphic to its image..

EXAMPLE 1.1.3. The following categories are abelian:

a) the category $A - mod$ of (say, left) A-modules over a given associative ring;

b) the category of sheaves of modules over a given sheaf of associative algebras.

The category of vector bundles *is not* an abelian category, because the quotient of two vector bundles is not a vector bundle in general.

1.2. Triangulated categories. Triangulated category is given by an additive category $\mathcal{C}$, a functor $[1] : \mathcal{C} \to \mathcal{C}$ called *translation (or shift) functor*, and a class of *distinguished triangles* $X \to Y \to Z$. These data satisfy a number of axioms, which will not be recalled here (see [Verdier]). The most complicated is the so-called octahedron axiom.

For a given abelian category $\mathcal{A}$ one constructs a triangulated category $D(\mathcal{A})$, called the *derived* category of $\mathcal{A}$. In fact $D(\mathcal{A})$ contains $\mathcal{A}$ as a full subcategory. The derived category $D(\mathcal{A})$ is obtained from the category of complexes in $\mathcal{A}$ by a kind of localization procedure. The shift functor [1] changes the grading of a complex: $([1](C))^i := (C[1])^i = C^{i+1}$. A morphism of two complexes, which induces an isomorphism on the cohomology, becomes an isomorphism in the derived category.

EXAMPLE 1.2.1. Let A be an associative algebra. We consider the category of complexes of *free* A-modules with morphisms given by the homotopy classes of morphisms of complexes. We get a triangulated category which is equivalent to $D^-(A - mod)$.

REMARK 1.2.2. If in the previous example one takes *all* A-modules, then the resulting category will not be equivalent to the derived category.

There are two kinds of triangulated categories:

a) algebraic;

b) topological.

An algebraic example is given by the derived category $D(\mathcal{C})$. Topological example is given by the category of homotopy types $\mathcal{HT}$. Objects of $\mathcal{HT}$ are pairs (X, n) where X is a CW-complex and $n \in \mathbf{Z}$. One defines $Hom((X, n), (Y, m)) = lim_{N\to\infty}[\Sigma^{N-n}X, \Sigma^{N-m}Y]$, where $[A, B]$ denotes the set of homotopy classes of maps, and Σ denotes the suspension functor. The shift functor is given by $[1] : (X, n) \mapsto (X, n + 1)$ (see [Switzer] for details).

1.3. Digression about non-commutative geometries. Here we would like to compare our ladder of categories with another ladder: the one of spaces.

1) A non-commutative space according to A. Connes is given by an associative algebra. The class of spaces (topological, smooth, etc.) is specified by a class of algebras (algebras of continuos functions, smooth functions, etc.).

2) A non-commutative space is given by an abelian category. This approach is based on the theorem (P. Gabriel, A. Rosenberg) that a scheme S can be reconstructed up to an isomorphism from the abelian category $Qcoh_S$ of quasi-coherent sheaves on S. Thus an abelian category should be interpreted as a category of quasi-coherent sheaves on a "non-commutative space".

3) A non-commutative space is given by a triangulated category. Main example of successful application of this approach will be discussed in the chapter devoted to

the mirror symmetry (in the terminology of physics: $N = 2$ Superconformal Field Theory).

Mathematically it is based on the desire to reconstruct a (projective) scheme X out of the category $D^b(X)$, the bounded derived category of coherent sheaves on X. It is known (A. Bondal, D. Orlov) that if X is of generic type (i.e. K_X or $-K_X$ is an ample sheaf), then such a reconstruction is unique up to an isomorphism. Nevertheless, it is not always the case: two non-isomorphic varieties X and Y can have equivalent categories $D^b(X) \simeq D^b(Y)$. This is typical for Calabi-Yau manifolds (for example, abelian varieties). We will discuss this topic later.

1.4. Why should one generalize triangulated categories? Triangulated categories were invented for purposes of homological algebra. They appear in a number of spectacular duality theorems. Nevertheless, this notion suffers from some deficiences. For example, the octahedron axiom is not motivated. It is not clear why one should not consider further axioms. Another problem is related to the notion of a cone of morphism.

Let $f : X \to Y$ be a morphism in a triangulated category $\mathcal{C}$. Then there is a object $C(f)$ called *a cone* of f, such that one has a distinguished triangle $X \to Y \to C(f)$. The cone $C(f)$ is not uniquely defined. Moreover, there is no functorial construction of the cone. More precisely, let us consider the following category $Mor(\mathcal{C})$. Objects of this category are morphisms in $\mathcal{C}$. Morphisms between $f : X \to Y$ and $f_1 : X_1 \to Y_1$ are pairs $\phi : X \to X_1, \psi : Y \to Y_1$ such that the natural diagram commutative. Then $f \to C(f)$ is not a functor from $G(\mathcal{C})$ to $\mathcal{C}$.

The following example demonstrates another problem. Let us consider the quiver A_2. Geometrically it is graph with two vertices and one directed arrow. Algebraically, it is given by a 3-dimensional algebra A_2 over a ground field $\mathbf{k}$, with a basis π_1, f, π_2 such that $\pi_i^2 = \pi_i, i = 1, 2$, and $\pi_1 + \pi_2 = 1$.

This algebra can be described also as an algebra related to the following category $\mathcal{C}$. The category $\mathcal{C}$ has two objects E, F. Morphisms are defined such as follows: $Hom(E, F) = \mathbf{k}$, $Hom(F, E) = 0$, $End(E) = End(F) = \mathbf{k}$.

Lemma 1.4.1. *For any* $\mathbf{k}$*-algebra* B *the category of* $B \otimes A_2$*-modules is equivalent to the category* $Mor(B - mod)$*.*

Proof. A representation of A_2 is given by a pair of k-vector spaces X, Y (they correspond to the idempotents $\pi_i, i = 1, 2$) as well as a linear morphism $f : X \to Y$ (it corresponds to $f \in A_2$). By definition the action of B commutes with the action of A_2. It follows that X and Y are B-invariant, and f is a homomorphism of B-modules. We leave to the reader the remaining details. $\square$

The previous example suggests the following idea. For a given abelian category $\mathcal{C}$ one should have an abelian category $\mathcal{C} \otimes A_2$ such that $Mor(\mathcal{C})$ is equivalent to $\mathcal{C} \otimes A_2$. If $\mathcal{C}$ is a triangulated category then $Mor(\mathcal{C})$ *is not* a triangulated category. Nevertheless in all known examples one can define a category $\mathcal{C} \otimes A_2$ such that it is a triangulated category, and the set of isomorphism classes of objects of $\mathcal{C} \otimes A_2$ is in one-to-one correspondence with isomorphism classes of objects in $Mor(\mathcal{C})$.

The conclusion is that some simple constructions fail to work in the case of triangulated categories. Therefore one should generalize this notion. An appropriate generalization will be discussed later in this chapter. It is called A_∞*-category.* Namely:

a) this is the "right" (from homotopical point of view) generalization of the notion of k-linear category;

b) triangulated A_∞-category generalizes the notion of usual triangulated category.

The relation between A_∞-categories and triangulated A_∞-categories is similar to the relation between *k-linear* and *additive* categories, rather than to the relation between abelian and derived categories. In other words, it is *simpler* than in the classical case. We are going to discuss A_∞-categories in detail in te second volume of the book. At the same time, main features of the theory can be observed in the case of A_∞-algebra, which can be thought of as A_∞-category with one object. In the Chapter, devoted to operads, we already met A_∞-algebras as algebras over the A_∞-operad (equivalently, algebras over the operad of singular chains on the collection of topological spaces $FM_1(n), n \geq 0$. In this chapter we use completely different point of view on A_∞-algebras. Namely, they will appear as local models for non-commutative formal pointed dg-manifolds. This makes the theory completely parallel to the theory of L_∞-algebras discussed before (which can be thought of as local models for *commutative* formal pointed dg-manifolds). At the same time, we decided not to combine both theories under the same roof. Indeed, the theory of commutative and non-commutative schemes, although having many similar features, are fundamentally different in problems and methods.

2. Coalgebras and non-commutative schemes

Geometric description of A_∞-algebras will be given in terms of geometry of non-commutative ind-affine schemes in the tensor category of graded vector spaces (we will use $\mathbf{Z}$-grading or $\mathbf{Z}/\mathbf{2}$-grading). In this section we are going to describe these ind-schemes as functors from finite-dimensional algebras to sets (cf. with the description of formal schemes in [Gr59]). More precisely, such functors are represented by counital coalgebras. Corresponding geometric objects are called *non-commutative thin schemes.*

2.1. Coalgebras as functors. Let k be a field, and $\mathcal{C}$ be a k-linear Abelian symmetric monoidal category (we call such categories *tensor*), which admits infinite sums and products. Then we can do simple linear algebra in $\mathcal{C}$, in particular, speak about associative algebras or coassociative coalgebras. By definition for a coalgebra B there is a morphism $\Delta : B \to B \otimes B$ called a *coproduct* (or comultiplication) such that $(\Delta \otimes id) \circ \Delta = (id \otimes \delta) \circ \Delta$. For a counital coalgebra B we also have a morphism $\varepsilon : B \to \mathbf{1}$, where $\mathbf{1}$ is the unit object in $\mathcal{C}$. The morphism ε is called a *counit* and satisfies the relation $(\varepsilon \otimes id) \circ \Delta = (id \otimes \varepsilon) \circ \Delta = id$.

Let $\Delta' = c_{B,B}\Delta$ denotes the opposite coproduct (here $c_{B,B} : B \otimes B \to B \otimes B$ is the commutativity morphism). The coalgebra is called *cocommutative* if $\Delta = \Delta'$. The *iterated coproduct* $\Delta^{(n)} : B^{\otimes n} \to B$ is defined by induction: $\Delta^{(2)} = \Delta$, $\Delta^{(n+1)} = (\Delta \otimes id^{\otimes n}) \circ \Delta^{(n)}$.

Definition 2.1.1. Non-counital coalgebra B is called conilpotent if there exists $n \geq 1$ such that $\Delta^{(n)} = 0$. It is called locally conilpotent if for any $b \in B$ there exists n (depending on b) such that $\Delta^{(n)}(b) = 0$.

If B is a counital coalgebras, we will keep the above terminology in the case when the non-counital coagebra $Ker\, \varepsilon \subset B$ is conilpotent (resp. locally conilpotent).

Clearly any conilpotent coalgebra is locally conilpotent.

For the rest of the Chapter, unless we say otherwise, we will assume that either $\mathcal{C} = Vect_k^{\mathbf{Z}}$, which is the tensor category of $\mathbf{Z}$-graded vector spaces $V = \oplus_{n\in\mathbf{Z}}V_n$, or $\mathcal{C} = Vect_k^{\mathbf{Z}/\mathbf{2}}$, which is the tensor category of $\mathbf{Z}/\mathbf{2}$-graded vector spaces (then $V = V_0 \oplus V_1$), or $\mathcal{C} = Vect_k$, which is the tensor category of k-vector spaces. Associativity morphisms in $Vect_k^{\mathbf{Z}}$ or $Vect_k^{\mathbf{Z}/\mathbf{2}}$ are identity maps, and commutativity morphisms are given by the Koszul rule of signs: $c(v_i \otimes v_j) = (-1)^{ij} v_j \otimes v_i$, where v_n denotes an element of degree n.

We will denote by $\mathcal{C}^f$ the Artinian category of finite-dimensional objects in $\mathcal{C}$ (i.e. objects of finite length). The category $Alg_{\mathcal{C}^f}$ of unital finite-dimensional algebras is closed with respect to finite projective limits. In particular, finite products and finite fiber products exist in $Alg_{\mathcal{C}^f}$. One has also the categories $Coalg_{\mathcal{C}}$ (resp. $Coalg_{\mathcal{C}^f}$) of coassociative counital (resp. coassociative counital finite-dimensional) coalgebras. In the case $\mathcal{C} = Vect_k$ we will also use the notation Alg_k, Alg_k^f, $Coalg_k$ and $Coalg_k^f$ for these categories. The category $Coalg_{\mathcal{C}^f} = Alg_{\mathcal{C}^f}^{op}$ admits finite inductive limits.

We will need simple facts about coalgebras. We will present proofs in the Appendix for completness.

THEOREM 2.1.2. *Let $F : Alg_{\mathcal{C}^f} \to Sets$ be a covariant functor commuting with finite projective limits. Then it is isomorphic to a functor of the type $A \mapsto Hom_{Coalg_{\mathcal{C}}}(A^*, B)$ for some counital coalgebra B. Moreover, the category of such functors is equivalent to the category of counital coalgebras.*

PROPOSITION 2.1.3. *If $B \in Ob(Coalg_{\mathcal{C}})$, then B is a union of finite-dimensional counital coalgebras.*

Objects of the category $Coalg_{\mathcal{C}^f} = Alg_{\mathcal{C}^f}^{op}$ can be interpreted as “very thin” non-commutative affine schemes (cf. with finite schemes in algebraic geometry). Proposition 1 implies that the category $Coalg_{\mathcal{C}}$ is naturally equivalent to the category of ind-objects in $Coalg_{\mathcal{C}^f}$.

For a counital coalgebra B we denote by $Spc(B)$ (the “spectrum” of the coalgebra B) the corresponding functor on the category of finite-dimensional algebras. A functor isomorphic to $Spc(B)$ for some B is called a *non-commutative thin scheme*. The category of non-commutative thin schemes is equivalent to the category of counital coalgebras. For a non-commutative scheme X we denote by B_X the corresponding coalgebra. We will call it the coalgebra of *distributions* on X. The *algebra of functions* on X is by definition $\mathcal{O}(X) = B_X^*$.

Non-commutative thin schemes form a full monoidal subcategory $NAff_{\mathcal{C}}^{th} \subset Ind(NAff_{\mathcal{C}})$ of the category of non-commutative ind-affine schemes (see Appendix).Tensor product corresponds to the tensor product of coalgebras.

Let us consider few examples.

EXAMPLE 2.1.4. Let $V \in Ob(\mathcal{C})$. Then $T(V) = \oplus_{n\geq 0} V^{\otimes n}$ carries a structure of counital cofree coalgebra in $\mathcal{C}$ with the coproduct $\Delta(v_0 \otimes ... \otimes v_n) = \sum_{0\leq i\leq n}(v_0 \otimes ... \otimes v_i) \otimes (v_{i+1} \otimes ... \otimes v_n)$. The corresponding non-commutative thin scheme is called non-commutative formal affine space V_{form} (or formal neighborhood of zero in V).

DEFINITION 2.1.5. A non-commutative formal manifold X is a non-commutative thin scheme isomorphic to some $Spc(T(V))$ from the example above. The dimension of X is defined as $dim_k V$.

The algebra $\mathcal{O}(\mathcal{X})$ of functions on a non-commutative formal manifold X of dimension n is isomorphic to the topological algebra $k\langle\langle x_1, ..., x_n\rangle\rangle$ of formal power series in free graded variables $x_1, ..., x_n$.

Let X be a non-commutative formal manifold, and $pt : k \to B_X$ a k-point in X,

DEFINITION 2.1.6. The pair (X, pt) is called a non-commutative formal pointed manifold. If $\mathcal{C} = Vect_k^{\mathbf{Z}}$ it will be called non-commutative formal pointed graded manifold. If $\mathcal{C} = Vect_k^{\mathbf{Z}/\mathbf{2}}$ it will be called non-commutative formal pointed supermanifold.

The following example is a generalization of the previous example (which corresponds to a quiver with one vertex).

EXAMPLE 2.1.7. Let I be a set and $B_I = \oplus_{i\in I}\mathbf{1}_i$ be the direct sum of trivial coalgebras. We denote by $\mathcal{O}(I)$ the dual topological algebra. It can be thought of as the algebra of functions on a discrete non-commutative thin scheme I.

A quiver Q in C with the set of vertices I is given by a collection of objects $E_{ij} \in \mathcal{C}, i, j \in I$ called spaces of arrows from i to j. The coalgebra of Q is the coalgebra B_Q generated by the $\mathcal{O}(I) - \mathcal{O}(I)$-bimodule $E_Q = \oplus_{i,j\in I}E_{ij}$, i.e. $B_Q \simeq \oplus_{n\geq 0}\oplus_{i_0,i_1,...,i_n\in I} E_{i_0i_1} \otimes ... \otimes E_{i_{n-1}i_n} := \oplus_{n\geq 0}B_Q^n$, $B_Q^0 := B_I$. Elements of B_Q^0 are called *trivial paths*. Elements of B_Q^n are called paths of the length n. Coproduct is given by the formula

$$\Delta(e_{i_0i_1} \otimes ... \otimes e_{i_{n-1}i_n}) = \oplus_{0\leq m\leq n}(e_{i_0i_1} \otimes ... \otimes e_{i_{m-1}i_m}) \otimes (e_{i_mi_{m+1}} ... \otimes ... \otimes e_{i_{n-1}i_n}),$$

where for $m = 0$ (resp. $m = n$) we set $e_{i_{-1}i_0} = 1_{i_0}$ (resp. $e_{i_ni_{n+1}} = 1_{i_n}$).

In particular, $\Delta(1_i) = 1_i \otimes 1_i, i \in I$ and $\Delta(e_{ij}) = 1_i \otimes e_{ij} + e_{ij} \otimes 1_j$, where $e_{ij} \in E_{ij}$, and $1_m \in B_I$ corresponds to the image of $1 \in \mathbf{1}$ under the natural embedding into $\oplus_{m\in I}\mathbf{1}$.

The coalgebra B_Q has a counit ε such that $\varepsilon(1_i) = 1_i$, and $\varepsilon(x) = 0$ for $x \in B_Q^n, n \geq 1$.

EXAMPLE 2.1.8. (Generalized quivers). Here we replace $\mathbf{1}_i$ by a unital simple algebra A_i (e.g. $A_i = Mat(n_i, D_i)$, where D_i is a division algebra). Then E_{ij} are $A_i - mod - A_j$-bimodules. We leave as an exercise to the reader to write down the coproduct (one uses the tensor product of bimodules) and to check that we indeed obtain a coalgebra.

EXAMPLE 2.1.9. Let I be a set. Then the coalgebra $B_I = \oplus_{i\in I}\mathbf{1}_i$ is a direct sum of trivial coalgebras, isomorphic to the unit object in $\mathcal{C}$. This is a special case of Example 2. Notice that in general B_Q is a $\mathcal{O}(I) - \mathcal{O}(I)$-bimodule.

EXAMPLE 2.1.10. Let A be an associative unital algebra. It gives rise to the functor $F_A : Coalg_{\mathcal{C}\{} \to Sets$ such that $F_A(B) = Hom_{Alg_{\mathcal{C}}}(A, B^*)$. This functor describes finite-dimensional representations of A. It commutes with finite direct limits, hence it is representable by a coalgebra. If $A = \mathcal{O}(X)$ is the algebra of regular functions on the affine scheme X, then in the case of algebraically closed field k the coalgebra representing F_A is isomorphic to $\oplus_{x\in X(k)}\mathcal{O}^*_{x,X}$, where $\mathcal{O}^*_{x,X}$ denotes the topological dual to the completion of the local ring $\mathcal{O}_{x,X}$. If X is smooth of dimension n, then each summand is isomorphic to the topological dual to the algebra of formal power series $k[[t_1,...,t_n]]$. In other words, this coalgebra corresponds to the disjoint union of formal neighborhoods of all points of X.

REMARK 2.1.11. One can describe non-commutative thin schemes more precisely by using structure theorems about finite-dimensional algebras in $\mathcal{C}$. For example, in the case $\mathcal{C} = Vect_k$ any finite-dimensional algebra A is isomorphic to a sum $A_0 \oplus r$, where A_0 is a finite sum of matrix algebras $\oplus_i Mat(n_i, D_i)$, D_i are division algebras, and r is the radical. In **Z**-graded case a similar decomposition holds, with A_0 being a sum of algebras of the type $End(V_i)\otimes D_i$,where V_i are some graded vector spaces and D_i are division algebras of degree zero. In **Z/2**-graded case the description is slightly more complicated. In particular A_0 can contain summands isomorphic to $(End(V_i)\otimes D_i)\otimes D_\lambda$, where V_i and D_i are **Z/2**-graded analogs of the above-described objects, and D_λ is a $1|1$-dimensional superalgebra isomorphic to $k[\xi]/(\xi^2=\lambda)$, $deg\,\xi = 1, \lambda\in k^*/(k^*)^2$.

2.2. Smooth thin schemes. Recall that the notion of an ideal has meaning in any abelian tensor category. A 2-sided ideal J is called *nilpotent* if the multiplication map $J^{\otimes n}\to J$ has zero image for a sufficiently large n.

DEFINITION 2.2.1. Counital coalgebra B in a tensor category $\mathcal{C}$ is called smooth if the corresponding functor $F_B : Alg_{\mathcal{C}^f}\to Sets, F_B(A) = Hom_{Coalg_\mathcal{C}}(A^*, B)$ satisfies the following lifting property: for any 2-sided nilpotent ideal $J\subset A$ the map $F_B(A)\to F_B(A/J)$ induced by the natural projection $A\to A/J$ is surjective. Non-commutative thin scheme X is called smooth if the corresponding counital coalgebra $B = B_X$ is smooth.

PROPOSITION 2.2.2. *For any quiver Q in $\mathcal{C}$ the corresponding coalgebra B_Q is smooth.*

Proof. First let us assume that the result holds for all finite quivers. We remark that if A is finite-dimensional, and Q is an infinite quiver then for any morphism $f : A^*\to B_Q$ we have: $f(A^*)$ belongs to the coalgebra of a finite sub-quiver of Q. Since the lifting property holds for the latter, the result follows. Finally, we need to prove the Proposition for a finite quiver Q . Let us choose a basis $\{e_{ij,\alpha}\}$ of each space of arrows E_{ij}. Then for a finite-dimensional algebra A the set $F_{B_Q}(A)$ is isomorphic to the set $\{((\pi_i), x_{ij,\alpha})_{i,j\in I}\}$, where $\pi_i\in A, \pi_i^2 = \pi_i, \pi_i\pi_j = \pi_j\pi_i$, if $i\neq j$,$\sum_{i\in I}\pi_i = 1_A$, and $x_{ij,\alpha}\in\pi_iA\pi_j$ satisfy the condition: there exists $N\geq 1$ such that $x_{i_1j_1,\alpha_1}...x_{i_mj_m,\alpha_m} = 0$ for all $m\geq N$. Let now $J\subset A$ be the nilpotent ideal from the definition of smooth coalgebra and $(\pi_i', x_{ij,\alpha}')$ be elements of A/J satisfying the above constraints. Our goal is to lift them to A. We can lift the them to the projectors π_i and elements $x_{ij,\alpha}$ for A in such a way that the above constraints are satisfied except of the last one, which becomes an inclusion $x_{i_1j_1,\alpha_1}...x_{i_mj_m,\alpha_m}\in J$ for $m\geq N$. Since $J^n = 0$ in A for some n we see that $x_{i_1j_1,\alpha_1}...x_{i_mj_m,\alpha_m} = 0$ in A for $m\geq nN$. This proves the result. ■

REMARK 2.2.3. a) According to Cuntz and Quillen (see [CQ95-2]) a non-commutative associative algebra R in $Vect_k$ is called *smooth* if the functor $Alg_k\to Sets, F_R(A) = Hom_{Alg_k}(R, A)$ satisfies the lifting property from the Definition 3 applied to all (not only finite-dimensional) algebras. We remark that if R is smooth in the sense of Cuntz and Quillen then the coalgebra R_{dual} representing the functor $Coalg_k^f \to Sets, B \mapsto Hom_{Alg_k^f}(R, B^*)$ is smooth. One can prove that any smooth coalgebra in $Vect_k$ is isomorphic to a coalgebra of a generalized quiver.

b) Almost all examples of non-commutative smooth thin schemes considered in this Chapter are non-commutative formal pointed manifolds, i.e. they are isomorphic to $Spc(T(V))$ for some $V \in Ob(\mathcal{C})$. It is natural to try to "globalize" this picture to the case of non-commutative "smooth" schemes X which satisfy the property that the completion of X at a "commutative" point gives rise to a formal pointed manifold in our sense. An example of the space of maps is considered below.

c) The tensor product of non-commutative smooth thin schemes is typically non-smooth, since it corresponds to the *tensor product* of coalgebras (the latter is not a categorical product).

Let now x be a k-point of a non-commutative smooth thin scheme X. By definition x is a homomorphism of counital coalgebras $x : k \to B_X$ (here $k = \mathbf{1}$ is the trivial coalgebra corresponding to the unit object). The completion $\widehat{X}_x$ of X at x is a formal pointed manifold which can be described such as follows. As a functor $F_{\widehat{X}_x} : Alg_{\mathcal{C}}^f \to Sets$ it assigns to a finite-dimensional algebra A the set of such homomorphisms of counital colagebras $f : A^* \to B_X$ which are compositions $A^* \to A_1^* \to B_X$, where $A_1^* \subset B_X$ is a conilpotent extension of x (i.e. A_1 is a finite-dimensional unital nilpotent algebra such that the natural embedding $k \to A_1^* \to B_X$ coinsides with $x : k \to B_X$).

Description of the coalgebra $B_{\widehat{X}_x}$ is given in the following Proposition.

PROPOSITION 2.2.4. *The formal neighborhood $\widehat{X}_x$ corresponds to the counital sub-coalgebra $B_{\widehat{X}_x} \subset B_X$ which is the preimage under the natural projection $B_X \to B_X/x(k)$ of the sub-coalgebra consisting of conilpotent elements in the non-counital coalgebra $B/x(k)$. Moreover, $\widehat{X}_x$ is universal for all morphisms from nilpotent extensions of x to X.*

We discuss in Appendix a more general construction of the completion along a non-commutative thin subscheme.

We leave as an exercise to the reader to prove the following result.

PROPOSITION 2.2.5. *Let Q be a quiver and $pt_i \in X = X_{B_Q}$ corresponds to a vertex $i \in I$. Then the formal neighborhood $\widehat{X}_{pt_i}$ is a formal pointed manifold corresponding to the tensor coalgebra $T(E_{ii}) = \oplus_{n \geq 0} E_{ii}^{\otimes n}$, where E_{ii} is the space of loops at i.*

2.3. Implicit and inverse function theorems. Here we give without proofs non-commutative analogs of the implicit and inverse function theorems. Proofs are basically the same as those in Chapter 3. We leave them as exercises to the reader.

THEOREM 2.3.1. *Let (X_1, pt_1) and (X_2, pt_2) be non-commutative formal pointed manifolds. Then a morphism $f : (X_1, pt_1) \to (X_2, pt_2)$ is an isomorphism if and only if the induced linear map of tangent spaces $f_1 = T(f) : T_{pt_1}(X_1) \to T_{pt_2}(X_2)$ is an isomorphism.*

THEOREM 2.3.2. *Let $f : (X_1, pt_1) \to (X_2, pt_2)$ be a morphism of non-commutative formal pointed manifolds such that the corresponding tangent map $f_1 : T_{pt_1}(X_1) \to$*

$T_{pt_2}(X_2)$ is an epimorphism. Then there exists a non-commutative formal pointed manifold (Y, pt_Y) such that $(X_1, pt_1) \simeq (X_2, pt_2) \times (Y, pt_Y)$, and under this isomorphism f becomes the natural projection.

If f_1 is a monomorphism, then there exists (Y, pt_Y) and an isomorphism $(X_2, pt_2) \to (X_1, pt_1) \times (Y, pt_Y)$, such that under this isomorphism f becomes the natural embedding $(X_1, pt_1) \to (X_1, pt_1) \times (pt_Y, pt_Y)$.

2.4. Inner Hom. Let X, Y be non-commutative thin schemes, and B_X, B_Y the corresponding coalgebras.

THEOREM 2.4.1. *The functor $Alg_{\mathcal{C}^f} \to Sets$ such that*

$$A \mapsto Hom_{Coalg_{\mathcal{C}}}(A^* \otimes B_X, B_Y)$$

is representable. The corresponding non-commutative thin scheme is denoted by $Maps(X, Y)$.

Proof. It is easy to see that the functor under consideration commutes with finite projective limits. Hence it is of the type $A \mapsto Hom_{Coalg_{\mathcal{C}}}(A^*, B)$, where B is a counital coalgebra The corresponding non-commutative thin scheme is the desired $Maps(X, Y)$. ■

It follows from the definition that $Maps(X, Y) = \underline{\mathrm{Hom}}(X, Y)$, where the inner Hom is taken in the symmetric monoidal category of non-commutative thin schemes. By definition $\underline{\mathrm{Hom}}(X, Y)$ is a non-commutative thin scheme, which satisfies the following functorial isomorphism for any $Z \in Ob(NAff_{\mathcal{C}}^{th})$:

$$Hom_{NAff_{\mathcal{C}}^{th}}(Z, \underline{\mathrm{Hom}}(X, Y)) \simeq Hom_{NAff_{\mathcal{C}}^{th}}(Z \otimes X, Y).$$

Notice that the monoidal category $NAff_{\mathcal{C}}$ of all non-commutative affine schemes does not have inner $Hom's$ even in the case $\mathcal{C} = Vect_k$. If $\mathcal{C} = Vect_k$ then one can define $\underline{\mathrm{Hom}}(X, Y)$ for $X = Spec(A)$, where A is a *finite-dimensional* unital algebra and Y is arbitrary. The situation is similar to the case of "commutative" algebraic geometry, where one can define an affine scheme of maps from a scheme of finite length to an arbitrary affine scheme. On the other hand, one can show that the category of non-commutative ind-affine schemes admits inner Hom's (the corresponding result for commutative ind-affine schemes is known.

REMARK 2.4.2. The non-commutative thin scheme $Maps(X, Y)$ gives rise to a quiver, such that its vertices are k-points of $Maps(X, Y)$. In other words, vertices correspond to homomorphisms $B_X \to B_Y$ of the coalgebras of distributions. Taking the completion at a k-point we obtain a formal pointed manifold. More generally, one can take a completion along a subscheme of k-points, thus arriving to a non-commutative formal manifold with a marked closed subscheme (rather than one point). This construction will be used in the second volume for the desription of the A_∞-structure on A_∞-functors. We also remark that the space of arrows E_{ij} of a quiver is an example of the geometric notion of bitangent space at a pair of k-points i, j. It will be also discussed in the second volume.

For non-counital coalgebras A and B we introduce a "new" tensor product $A \otimes^{new} B = A \otimes B \oplus A \oplus B$. It mimicks the tensor product of counital coalgebras. Then the functor on non-unital finite-dimensional coalgebras

$$C \mapsto Hom_{Coalg_{\mathcal{C}}}(C \otimes^{new} A, B)$$

is representable by a non-counital coalgebra, which can be thought of the coalgebra of distributions on a thin scheme $Maps(Spc(A), Spc(B))$ corresponding to non-countial coalgebras A and B.

EXAMPLE 2.4.3. Let $Q_1 = \{i_1\}$ and $Q_2 = \{i_2\}$ be quivers with one vertex such that $E_{i_1 i_1} = V_1, E_{i_2 i_2} = V_2$, $dim\, V_i < \infty, i = 1, 2$. Then $B_{Q_i} = T(V_i), i = 1, 2$ and $Maps(X_{B_{Q_1}}, X_{B_{Q_2}})$ corresponds to the quiver Q such that the set of vertices $I_Q = Hom_{Coalg_{\mathcal{C}}}(B_{Q_1}, B_{Q_2}) = \prod_{n\geq 1} \underline{\mathrm{Hom}}(V_1^{\otimes n}, V_2)$ and for any two vertices $f, g \in I_Q$ the space of arrows is isomorphic to $E_{f,g} = \prod_{n\geq 0} \underline{\mathrm{Hom}}(V_1^{\otimes n}, V_2)$.

DEFINITION 2.4.4. Homomorphism $f : B_1 \to B_2$ of counital coalgebras is called a minimal conilpotent extension if it is an inclusion and the induced coproduct on the non-counital coalgebra $B_2/f(B_1)$ is trivial.

Composition of minimal conilpotent extensions is simply called a conilpotent extension. Definition 2.2.1 can be reformulated in terms of finite-dimensional coalgebras. Coalgebra B is smooth if the functor $C \mapsto Hom_{Coalg_{\mathcal{C}}}(C, B)$ satisfies the lifting property with respect to conilpotent extensions of finite-dimensional counital coalgebras. The following proposition shows that we can drop the condition of finite-dimensionality.

PROPOSITION 2.4.5. *If B is a smooth coalgebra then the functor $Coalg_{\mathcal{C}} \to Sets$ such that $C \mapsto Hom_{Coalg_{\mathcal{C}}}(C, B)$ satisfies the lifting property for conilpotent extensions.*

Proof. Let $f : B_1 \to B_2$ be a conilpotent extension, and $g : B_1 \to B$ and arbitrary homomorphism of counital coalgebras. It can be thought of as homomorphism of $f(B_1) \to B$. We need to show that g can be extended to B_2. Let us consider the set of pairs (C, g_C) such $f(B_1) \subset C \subset B_2$ and $g_C : C \to B$ defines an extension of counital coalgebras, which coincides with g on $f(B_1)$. We apply Zorn lemma to the partially ordered set of such pairs and see that there exists a maximal element (B_{max}, g_{max}) in this set. We claim that $B_{max} = B_2$. Indeed, let $x \in B_2 \setminus B_{max}$. Then there exists a finite-dimensional coalgebra $B_x \subset B_2$ which contains x. Clearly B_x is a conilpotent extension of $f(B_1) \cap B_x$. Since B is smooth we can extend $g_{max} : f(B_1) \cap B_x \to B$ to $g_x : B_x \to B$ and,finally to $g_{x,max} : B_x + B_{max} \to B$. This contradicts to maximality of (B_{max}, g_{max}). Proposition is proved. ■

PROPOSITION 2.4.6. *If X, Y are non-commutative thin schemes and Y is smooth then $Maps(X, Y)$ is also smooth.*

Proof. Let $A \to A/J$ be a nilpotent extension of finite-dimensional unital algebras. Then $(A/J)^* \otimes B_X \to A^* \otimes B_X$ is a conilpotent extension of counital coalgebras. Since B_Y is smooth then the previous Proposition implies that the induced map $Hom_{Coalg_{\mathcal{C}}}(A^* \otimes B_X, B_Y) \to Hom_{Coalg_{\mathcal{C}}}((A/J)^* \otimes B_X, B_Y)$ is surjective. This concludes the proof. ■

Let us consider the case when (X, pt_X) and (Y, pt_Y) are non-commutative formal pointed manifolds in the category $\mathcal{C} = Vect_k^{\mathbf{Z}}$. One can describe "in coordinates" the non-commutative formal pointed manifold, which is the formal neighborhood of a k-point of $Maps(X, Y)$. Namely, let $X = Spc(B)$ and $Y = Spc(C)$, and let $f \in Hom_{NAff_{\mathcal{C}}^{th}}(X, Y)$ be a morphism preserving marked points. Then f gives rise to a k-point of $Z = Maps(X, Y)$. Since $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are isomorphic to the topological algebras of formal power series in free graded variables, we can choose sets of free topological generators $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ for these algebras. Then we can write for the corresponding homomorphism of algebras $f^*:\mathcal{O}(Y)\to\mathcal{O}(X)$:

$$f^*(y_j)=\sum_I c^0_{j,M}x^M,$$

where $c^0_{j,M}\in k$ and $M=(i_1,...,i_n), i_s\in I$ is a non-commutative multi-index (all the coefficients depend on f, hence a better notation should be $c^{f,0}_{j,M}$). Notice that for $M=0$ one gets $c^0_{j,0}=0$ since f is a morphism of pointed schemes. Then we can consider an "infinitesimal deformation" f_{def} of f

$$f^*_{def}(y_j)=\sum_M (c^0_{j,M}+\delta c^0_{j,M})x^M,$$

where $\delta c^0_{j,M}$ are *new variables* commuting with all x_i. Then $\delta c^0_{j,M}$ can be thought of as coordinates in the formal neighborhood of f. More pedantically it can be spelled out such as follows. Let $A=k\oplus m$ be a finite-dimensional graded unital algebra, where m is a graded nilpotent ideal of A. Then an A-point of the formal neighborhood U_f of f is a morphism $\phi\in Hom_{NAff^{th}_{\mathcal{C}}}(Spec(A)\otimes X,Y)$, such that it reduces to f modulo the nilpotent ideal m. We have for the corresponding homomorphism of algebras:

$$\phi^*(y_j)=\sum_M c_{j,M}x^M,$$

where M is a non-commutative multi-index, $c_{j,M}\in A$, and $c_{j,M}\mapsto c^0_{j,M}$ under the natural homomorphism $A\to k=A/m$. In particular $c_{j,0}\in m$. We can treat coefficients $c_{j,M}$ as A-points of the formal neighborhood U_f of $f\in Maps(X,Y)$.

REMARK 2.4.7. The above definitions will play an important role in the subsequent paper, where the non-commutative smooth thin scheme $Spc(B_Q)$ will be assigned to a (small) A_∞-category, the non-commutative smooth thin scheme

$Maps(Spc(B_{Q_1}),Spc(B_{Q_2}))$ will be used for the description of the category of A_∞-functors between A_∞-categories, and the formal neighborhood of a point in the space $Maps(Spc(B_{Q_1}),Spc(B_{Q_2}))$ will correspond to natural transformations between A_∞-functors.

REMARK 2.4.8. Let $A=B, f=id$. Then $Spc(Maps(Spc(A),Spc(A))$ is a Hopf algebra. It might be interesting to study it further.

We conclude this subsection with the following result.

PROPOSITION 2.4.9. *There is natural isomorphism of Lie algebras* $Vect(X)\simeq T_{id_X}(\widehat{Maps}_{id_X}(X,X))$.

Proof. Tangent space in the RHS can be identified with automorphisms of the coalgebra $\underline{\mathrm{Hom}}_{id}(C,C)\otimes k[t]/(t^2)$. Equivalently it is a set of maps from the coalgebra $Spc((k[t]/(t^2))^*)$ to X. But the latter is the set of all vector fields on X. We leave to the reader to check that in fact we have an isomorphism of local groups, which implies an isomorphism of Lie algebras. ■

3. A_∞-algebras

3.1. Main definitions. From now on assume that $\mathcal{C} = Vect_k^{\mathbf{Z}}$ unless we say otherwise. If X is a thin scheme then a vector field on X is, by definition, a derivation of the coalgebra B_X. Vector fields form a graded Lie algebra $Vect(X)$.

DEFINITION 3.1.1. A non-commutative thin differential-graded (dg for short) scheme is a pair (X, d) where X is a non-commutative thin scheme, and d is a vector field on X of degree $+1$ such that $[d, d] = 0$.

We will call the vector field d *homological vector field.*

Let X be a formal pointed manifold and x_0 be its unique k-point. Such a point corresponds to a homomorphism of counital coalgebras $k \to B_X$. We say that the vector field d vanishes at x_0 if the corresponding derivation kills the image of k.

DEFINITION 3.1.2. A non-commutative formal pointed dg-manifold is a pair $((X, x_0), d)$ such that (X, x_0) is a non-commutative formal pointed graded manifold, and $d = d_X$ is a homological vector field on X such that $d|_{x_0} = 0$.

Homological vector field d has an infinite Taylor decomposition at x_0. More precisely, let $T_{x_0}X$ be the tangent space at x_0. It is canonically isomorphic to the graded vector space of primitive elements of the coalgebra B_X, i.e. the set of $a \in B_X$ such that $\Delta(a) = 1 \otimes a + a \otimes 1$ where $1 \in B_X$ is the image of $1 \in k$ under the homomorphism of coalgebras $x_0 : k \to B_X$ (see Appendix for the general definition of the tangent space). Then $d := d_X$ gives rise to a (non-canonically defined) collection of linear maps $d_X^{(n)} := m_n : T_{x_0}X^{\otimes n} \to T_{x_0}X[1], n \geq 1$ called *Taylor coefficients of* d which satisfy a system of quadratic relations arising from the condition $[d, d] = 0$. Indeed, our non-commutative formal pointed manifold is isomorphic to the formal neighborhood of zero in $T_{x_0}X$, hence the corresponding non-commutative thin scheme is isomorphic to the cofree tensor coalgebra $T(T_{x_0}X)$ generated by $T_{x_0}X$. Homological vector field d is a derivation of a cofree coalgebra, hence it is uniquely determined by a sequence of linear maps m_n.

DEFINITION 3.1.3. Non-unital A_∞-algebra over k is given by a non-commutative formal pointed dg-manifold (X, x_0, d) together with an isomorphism of counital coalgebras $B_X \simeq T(T_{x_0}X)$.

Choice of an isomorphism with the tensor coalgebra generated by the tangent space is a non-commutative analog of a choice of affine structure in the formal neighborhood of x_0.

From the above definitions one can recover the traditional one. We present it below for convenience of the reader.

DEFINITION 3.1.4. A structure of an A_∞-algebra on $V \in Ob(Vect_k^{\mathbf{Z}})$ is given by a derivation d of degree $+1$ of the non-counital cofree coalgebra $T_+(V[1]) = \oplus_{n\geq 1} V^{\otimes n}$ such that $[d, d] = 0$ in the differential-graded Lie algebra of coalgebra derivations.

Traditionally the Taylor coefficients of $d = m_1 + m_2 + \ldots$ are called (higher) multiplications for V. The pair (V, m_1) is a complex of k-vector spaces called the *tangent complex.* If $X = Spc(T(V))$ then $V[1] = T_0X$ and $m_1 = d_X^{(1)}$ is the first Taylor coefficient of the homological vector field d_X. The tangent cohomology groups $H^i(V, m_1)$ will be denoted by $H^i(V)$. Clearly $H^\bullet(V) = \oplus_{i\in\mathbf{Z}} H^i(V)$ is an associative (non-unital) algebra with the product induced by m_2.

An important class of A_∞-algebras consists of *unital* (or strictly unital) and *weakly unital* (or homologically unital) ones. We are going to discuss the definition and the geometric meaning of unitality later.

Homomorphism of A_∞-algebras can be described geometrically as a morphism of the corresponding non-commutative formal pointed dg-manifolds. In the algebraic form one recovers the following traditional definition.

DEFINITION 3.1.5. A homomorphism of non-unital A_∞-algebras (A_∞-morphism for short) $(V, d_V) \to (W, d_W)$ is a homomorphism of differential-graded coalgebras $T_+(V[1]) \to T_+(W[1])$.

A homomorphism f of non-unital A_∞-algebras is determined by its Taylor coefficients $f_n : V^{\otimes n} \to W[1-n], n \geq 1$ satisfying the system of equations

$\sum_{1 \leq l_1 < ..., < l_i = n} (-1)^{\gamma_i} m_i^W (f_{l_1}(a_1, ..., a_{l_1}),$
$f_{l_2 - l_1}(a_{l_1+1}, ..., a_{l_2}), ..., f_{n-l_{i-1}}(a_{n-l_{i-1}+1}, ..., a_n)) =$
$\sum_{s+r=n+1} \sum_{1 \leq j \leq s} (-1)^{\epsilon_s} f_s(a_1, ..., a_{j-1}, m_r^V(a_j, ..., a_{j+r-1}), a_{j+r}, ..., a_n).$

Here $\epsilon_s = r \sum_{1 \leq p \leq j-1} deg(a_p) + j - 1 + r(s-j)$, $\gamma_i = \sum_{1 \leq p \leq i-1} (i-p)(l_p - l_{p-1} - 1) + \sum_{1 \leq p \leq i-1} \nu(l_p) \sum_{l_{p-1}+1 \leq q \leq l_p} deg(a_q)$, where we use the notation $\nu(l_p) = \sum_{p+1 \leq m \leq i} (1 - l_m + l_{m-1})$, and set $l_0 = 0$.

REMARK 3.1.6. All the above definitions and results are valid for $\mathbf{Z}/\mathbf{2}$-graded A_∞-algebras as well. In this case we consider formal manifolds in the category $Vect_k^{\mathbf{Z}/\mathbf{2}}$ of $\mathbf{Z}/\mathbf{2}$-graded vector spaces. We will use the correspodning results without further comments. In this case one denotes by ΠA the $\mathbf{Z}/\mathbf{2}$-graded vector space $A[1]$.

3.2. Minimal models of A_∞-algebras. One can do simple differential geometry in the symmetric monoidal category of non-commutative formal pointed dg-manifolds. New phenomenon is the possibility to define some structures up to a quasi-isomorphism.

DEFINITION 3.2.1. Let $f : (X, d_X, x_0) \to (Y, d_Y, y_0)$ be a morphism of non-commutative formal pointed dg-manifolds. We say that f is a quasi-isomorphism if the induced morphism of the tangent complexes $f_1 : (T_{x_0}X, d_X^{(1)}) \to (T_{y_0}Y, d_Y^{(1)})$ is a quasi-isomorphism. We will use the same terminology for the corresponding A_∞-algebras.

DEFINITION 3.2.2. An A_∞-algebra A (or the corresponding non-commutative formal pointed dg-manifold) is called minimal if $m_1 = 0$. It is called contractible if $m_n = 0$ for all $n \geq 2$ and $H^\bullet(A, m_1) = 0$.

The notion of minimality is coordinate independent, while the notion of contractibility is not.

It is easy to prove that any A_∞-algebra A has a *minimal model* M_A, i.e. M_A is minimal and there is a quasi-isomorphism $M_A \to A$ (the proof is similar to the one in Chapter 3. The minimal model is unique up to an A_∞-isomorphism. We will use the same terminology for non-commutative formal pointed dg-manifolds. In geometric language a non-commutative formal pointed dg-manifold X is isomorphic to a *categorical* product (i.e. corresponding to the completed free product of algebras of functions) $X_m \times X_{lc}$, where X_m is minimal and X_{lc} is linear contractible. The above-mentioned quasi-isomorphism corresponds to the projection $X \to X_m$.

The following result (homological inverse function theorem) can be easily deduced from the above product decomposition.

PROPOSITION 3.2.3. *If $f : A \to B$ is a quasi-isomorphism of A_∞-algebras then there is a (non-canonical) quasi-isomorphism $g : B \to A$ such that fg and gf induce identity maps on zero cohomologies $H^0(B)$ and $H^0(A)$ respectively.*

3.3. A_∞-algebra structure on a subcomplex. Let $(A, m_n), n \geq 1$ be a non-unital A_∞-algebra, $\Pi : A \to A$ be an idempotent which commutes with the differential $d = m_1$. In other words, Π is a linear map of degree zero such that $d\Pi = \Pi d, \Pi^2 = \Pi$. Assume that we are given an homotopy $H : A \to A[-1]$, $1 - \Pi = dH + Hd$ where 1 denotes the identity morphism. Let us denote the image of Π by B. Then we have an embedding $i : B \to A$ and a projection $p : A \to B$, such that $\Pi = i \circ p$.

Let us introduce a sequence of linear operations $m_n^B : B^{\otimes n} \to B[2-n]$ in the following way:

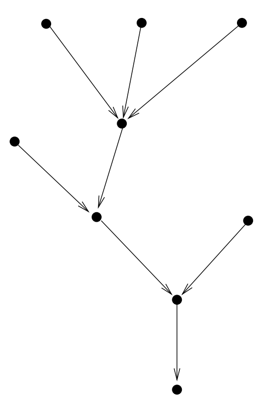
a) $m_1^B := d^B = p \circ m_1 \circ i$;

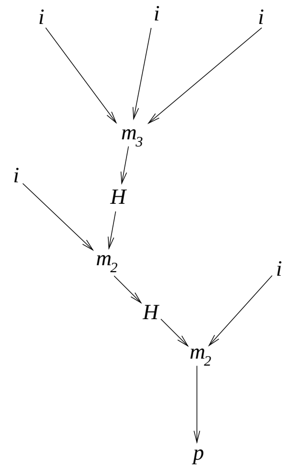
b) $m_2^B = p \circ m_2 \circ (i \otimes i)$;

c) $m_n^B = \sum_T \pm m_{n,T}, n \geq 3$.

Here the summation is taken over all oriented planar trees T with $n+1$ tails vertices (including the root vertex), such that the (oriented) valency $|v|$ (the number of ingoing edges) of every internal vertex of T is at least 2. In order to describe the linear map $m_{n,T} : B^{\otimes n} \to B[2-n]$ we need to make some preparations. Let us consider another tree $\bar{T}$ which is obtained from T by the insertion of a new vertex into every internal edge. As a result, there will be two types of internal vertices in $\bar{T}$: the "old" vertices, which coincide with the internal vertices of T, and the "new" ones, which can be thought geometrically as the midpoints of the internal edges of T.

To every tail vertex of $\bar{T}$ we assign the embedding i. To every "old" vertex v we assign m_k with $k = |v|$. To every "new" vertex we assign the homotopy operator H. To the root we assign the projector p. Then moving along the tree down to the root one reads off the map $m_{n,T}$ as the composition of maps assigned to vertices of $\bar{T}$. Here is an example of T and $\bar{T}$:





PROPOSITION 3.3.1. *The linear map m_1^B defines a differential in B.*

Proof. Clear. ■

THEOREM 3.3.2. *The sequence $m_n^B, n \geq 1$ gives rise to a structure of an A_∞-algebra on B.*

Sketch of the proof. The proof is quite straightforward, so we just briefly show main steps of computations.

First, one observes that p and i are homomorphisms of complexes. In order to prove the theorem we will replace for a given $n \geq 2$ each summand $m_{n,T}$ by a different one, and then compute the result in two different ways. Let us consider a collection of trees $\{\bar{T}_e\}_{e \in E(\bar{T})}$ such that $\bar{T}_e$ is obtained from $\bar{T}$ in the following way:

a) we split the edge e into two edges by inserting a new vertex w_e inside e;

b) the remaining part of $\bar{T}$ is unchanged.

We assign $d = m_1$ to the vertex w_e edge, and keep all other assignments untouched. In this way we obtain a map $m_{n,\bar{T}_e} : B^{\otimes n} \to B[3-n]$.

Let us consider the following sum:

$$\hat{m}_n^B = \sum_T \sum_{e \in E(\bar{T})} \pm m_{n,\bar{T}_e}.$$

VPISAT ZNAKI?

We can compute it in two different ways: using the relation $1 - \Pi = dH + Hd$, and using the formulas for $d(m_j)$, $j \geq 2$ given by the A_∞-structure on A. The case of the relation $1 - \Pi = dH + Hd =: d(H)$ gives

$$\hat{m}_n^B = d(m_n^B) - m_n^{B,\Pi} + m_n^{B,1}$$

where $m_n^{B,\Pi}$ is defined analogously to m_n^B, with the only difference that we assign to a new vertex operator Π instead of H for some edge $e \in E_i(T)$. Similarly, the summand $m_n^{B,1}$ is defined if we assign to a new vertex operator $1 = id_A$ instead of H. Formulas for $d(m_j)$ are quadratic expressions in m_l, $l < j$. This gives us another identity

$$\hat{m}_n^B = m_n^{B,1}$$

Thus we have $d(m_n^B) = m_n^{B,\Pi}$, and it is exactly the A_∞-constraint for the collection $(m_n^B)_{n\geq 1}$. ∎

Moreover, using similar technique, one can prove the following result.

PROPOSITION 3.3.3. *There is a canonical A_∞-morphism $g : B \to A$, which defines a quasi-isomorphism of A_∞-algebras.*

For the convenience fo the reader we give an explicit formula for a canonical choice of g. The operator $g_1 : B \to A$ is defined as the inclusion i. For $n \geq 2$ we define g_n as the sum of terms $g_{n,T}$ over all planar trees T with $n+1$ tails. Each term $g_{n,T}$ is similar to the term $m_{n,T}$ defined above, the only difference is that we insert operator H instead of p into the root vertex.

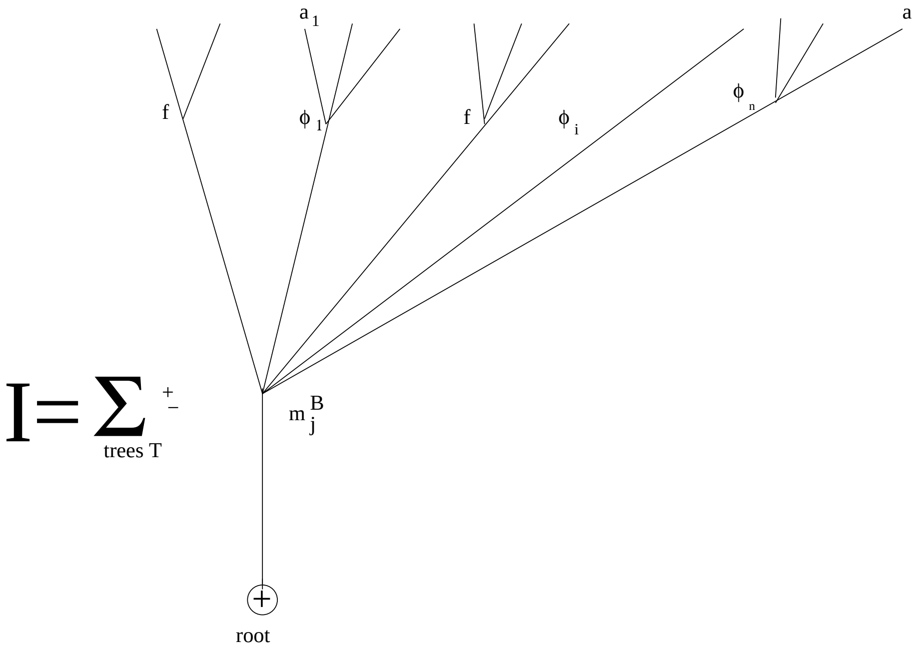
One can also construct an explicit A_∞-quasi-isomorphism $A \to B$.

3.4. Centralizer of an A_∞-morphism. Let A and B be two A_∞-algebras, and (X, d_X, x_0) and (Y, d_Y, y_0) be the corresponding non-commutative formal pointed dg-manifolds. Let $f : A \to B$ be a morphism of A_∞-algebras. Then the corresponding k-point $f \in Maps(Spc(A), Spc(B))$ gives rise to the formal pointed manifold $U_f = \widehat{Maps}(X, Y)_f$ (completion at the point f). Functoriality of the construction of $Maps(X, Y)$ gives rise to a homomorphism of graded Lie algebras of vector fields $Vect(X) \oplus Vect(Y) \to Vect(Maps(X, Y))$. Since $[d_X, d_Y] = 0$ on $X \otimes Y$, we have a well-defined homological vector field d_Z on $Z = Maps(X, Y)$. It corresponds to $d_X \otimes 1_Y - 1_X \otimes d_Y$ under the above homomorphism. It is easy to see that $d_Z|_f = 0$ and in fact morphisms $f : A \to B$ of A_∞-algebras are exactly zeros of d_Z. We are going to describe below the A_∞-algebra $Centr(f)$ (centralizer of f) which corresponds to the formal neighborhood U_f of the point $f \in Maps(X, Y)$. We can write (see Section 2.3 for the notation)

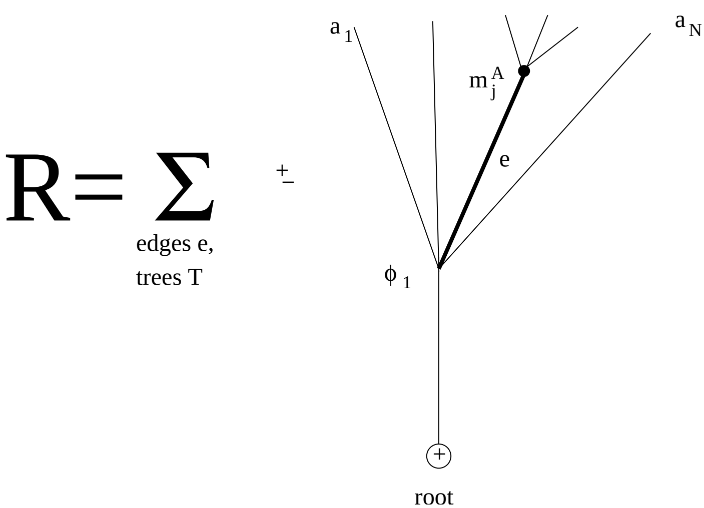
$$c_{j,M} = c_{j,M}^0 + r_{j,M},$$

where $c_{j,M}^0 \in k$ and $r_{j,M}$ are formal non-commutative coordinates in the neighborhood of f. Then the A_∞-algebra $Centr(f)$ has a basis $(r_{j,M})_{j,M}$ and the A_∞-structure is defined by the restriction of the homological vector d_Z to U_f.

As a **Z**-graded vector space $Centr(f) = \prod_{n\geq 0} Hom_{Vect_k^{\mathbf{Z}}}(A^{\otimes n}, B)[-n]$. Let $\phi_1, ..., \phi_n \in Centr(f)$, and $a_1, ..., a_N \in A$. Then we have $m_n(\phi_1, ..., \phi_n)(a_1, ..., a_N) = I + R$. Here I corresponds to the term $= 1_X \otimes d_Y$ and is given by the following expression



Similarly R corresponds to the term $d_X \otimes 1_Y$ and is described by the following picture



Comments on the figure describing I.

1) We partition a sequence $(a_1, ..., a_N)$ into $l \geq n$ non-empty subsequences.

2) We mark n of these subsequences counting from the left (the set can be empty).

3) We apply multilinear map $\phi_i, 1 \leq i \leq n$ to the ith marked group of elements a_l.

4) We apply Taylor coefficients of f to the remaining subsequences.

Notice that the term R appear only for m_1 (i.e. $n = 1$). For all subsequences we have $n \geq 1$.

From geometric point of view the term I corresponds to the vector field d_Y, while the term R corresponds to the vector field d_X.

PROPOSITION 3.4.1. *Let $d_{Centr(f)}$ be the derivation corresponding to the image of $d_X \oplus d_Y$ in $Maps(X, Y)$.*

One has $[d_{Centr(f)}, d_{Centr(f)}] = 0$.

Proof. Clear. ■

REMARK 3.4.2. The A_∞-algebra $Centr(f)$ and its generalization to the case of A_∞-categories discussed in the second volume give geometric description of the notion of natural transformaion between A_∞-functors.

4. Non-commutative dg-line L and weak unit

4.1. Main definition.

DEFINITION 4.1.1. An A_∞-algebra is called *unital* (or strictly unital) if there exists an element $1 \in V$ of degree zero, such that $m_2(1, v) = m_2(v, 1)$ and $m_n(v_1, ..., 1, ..., v_n) = 0$ for all $n \neq 2$ and $v, v_1, ..., v_n \in V$. It is called *weakly unital* (or homologically unital) if the graded associative unital algebra $H^\bullet(V)$ has a unit $1 \in H^0(V)$.

The notion of strict unit depends on a choice of affine coordinates on $Spc(T(V))$, while the notion of weak unit is "coordinate free". Moreover, one can show that a weakly unital A_∞-algebra becomes strictly unital after an appropriate change of coordinates.

The category of unital or weakly unital A_∞-algebras are defined in the natural way by the requirement that morphisms preserve the unit (or weak unit) structure.

In this section we are going to discuss a non-commutative dg-version of the odd 1-dimensional supervector space $\mathbf{A}^{0|1}$ and its relationship to weakly unital A_∞-algebras. The results are valid for both $\mathbf{Z}$-graded and $\mathbf{Z/2}$-graded A_∞-algebras.

DEFINITION 4.1.2. Non-commutative formal dg-line $\mathbf{L}$ is a non-commutative formal pointed dg-manifold corresponding to the one-dimensional A_∞-algebra $A \simeq k$ such that $m_2 = id, m_{n \neq 2} = 0$.

The algebra of functions $\mathcal{O}(\mathbf{L})$ is isomorphic to the topological algebra of formal series $k\langle\langle\xi\rangle\rangle$, where $deg\, \xi = 1$. The differential is given by $\partial(\xi) = \xi^2$.

4.2. Adding a weak unit. Let (X, d_X, x_0) be a non-commutative formal pointed dg-manifold correspodning to a non-unital A_∞-algebra A. We would like to describe geometrically the procedure of adding a weak unit to A.

Let us consider the non-commutative formal pointed graded manifold $X_1 = \mathbf{L} \times X$ corresponding to the free product of the coalgebras $B_\mathbf{L} * B_X$. Clearly one can lift vector fields d_X and $d_\mathbf{L} := \partial/\partial\, \xi$ to X_1.

LEMMA 4.2.1. *The vector field*

$$d := d_{X_1} = d_X + ad(\xi) - \xi^2 \partial/\partial\, \xi$$

satisfies the condition $[d, d] = 0$.

Proof. Straightforward check. ∎

It follows from the formulas given in the proof that ξ appears in the expansion of d_X in quadratic expressions only. Let A_1 be an A_∞-algebras corresponding to X_1 and $1 \in T_{pt}X_1 = A_1[1]$ be the element of $A_1[1]$ dual to ξ (it corresponds to the tangent vector $\partial/\partial\xi$). Thus we see that $m_2^{A_1}(1,a) = m_2^{A_1}(a,1) = a, m_2^{A_1}(1,1) = 1$ for any $a \in A$ and $m_n^{A_1}(a_1,...,1,...,a_n) = 0$ for all $n \geq 2, a_1,...,a_n \in A$. This proves the following result.

PROPOSITION 4.2.2. *The A_∞-algebra A_1 has a strict unit.*

Notice that we have a canonical morphism of non-commutative formal pointed dg-manifolds $e : X \to X_1$ such that $e^*|_X = id, e^*(\xi) = 0$.

DEFINITION 4.2.3. Weak unit in X is given by a morphism of non-commutative formal pointed dg-manifolds $p : X_1 \to X$ such that $p \circ e = id$.

It follows from the definition that if X has a weak unit then the associative algebra $H^\bullet(A, m_1^A)$ is unital. Hence our geometric definition agrees with the pure algebraic one (explicit algebraic description of the notion of weak unit can be found e.g. in [FOOO], Section 20).

5. Modules and bimodules

5.1. Modules and vector bundles. Recall that a topological vector space is called linearly compact if it is a projective limit of finite-dimensional vector spaces. The duality functor $V \mapsto V^*$ establishes an anti-equivalence between the category of vector spaces (equipped with the discrete topology) and the category of linearly compact vector spaces. All that can be extended in the obvious way to the category of graded vector spaces.

Let X be a non-commutative thin scheme in $Vect_k^{\mathbf{Z}}$.

DEFINITION 5.1.1. Linearly compact vector bundle $\mathcal{E}$ over X is given by a linearly compact topologically free $\mathcal{O}(X)$-module $\Gamma(\mathcal{E})$, where $\mathcal{O}(X)$ is the algebra of function on X. Module $\Gamma(\mathcal{E})$ is called the module of *sections* of the linearly compact vector bundle $\mathcal{E}$.

Suppose that (X, x_0) is formal graded manifold. *The fiber* of $\mathcal{E}$ over x_0 is given by the quotient space $\mathcal{E}_{x_0} = \Gamma(\mathcal{E})/\overline{m_{x_0}\Gamma(\mathcal{E})}$ where $m_{x_0} \subset \mathcal{O}(X)$ is the 2-sided maximal ideal of functions vanishing at x_0, and the bar means the closure.

DEFINITION 5.1.2. A dg-vector bundle over a formal pointed dg-manifold (X, d_X, x_0) is given by a linearly compact vector bundle $\mathcal{E}$ over (X, x_0) such that the corresponding module $\Gamma(\mathcal{E})$ carries a differential $d_\mathcal{E} : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})[1], d_\mathcal{E}^2 = 0$ so that $(\Gamma(\mathcal{E}), d_\mathcal{E})$ becomes a dg-module over the dg-algebra $(\mathcal{O}(X), d_X)$ and $d_\mathcal{E}$ vanishes on $\mathcal{E}_{x_0}$.

DEFINITION 5.1.3. Let A be a non-unital A_∞-algebra. A left A-module M is given by a dg-bundle E over the formal pointed dg-manifold $X = Spc(T(A[1]))$ together with an isomorphism of vector bundles $\Gamma(\mathcal{E}) \simeq \mathcal{O}(X)\widehat{\otimes}M^*$ called a trivialization of $\mathcal{E}$.

Passing to dual spaces we obtain the following algebraic definition.

DEFINITION 5.1.4. Let A be an A_∞-algebra and M be a $\mathbf{Z}$-graded vector space. A structure of a left A_∞-module on M over A (or simply a structure of a left A-module on M) is given by a differential d_M of degree $+1$ on $T(A[1]) \otimes M$ which makes it into a dg-comodule over the dg-coalgebra $T(A[1])$.

The notion of *right* A_∞-module is similar. Right A-module is the same as left A^{op}-module. Here A^{op} is the *opposite* A_∞-algebra, which coincides with A as a $\mathbf{Z}$-graded vector space, and for the higher multiplications one has: $m_n^{op}(a_1, ..., a_n) = (-1)^{n(n-1)/2} m_n(a_n, ..., a_1)$. The A_∞-algebra A carries the natural structures of the left and right A-modules. If we simply say "A-module" it will always mean "left A-module".

Taking the Taylor series of d_M we obtain a collection of k-linear maps (higher action morphisms) for any $n \geq 1$

$$m_n^M : A^{\otimes(n-1)} \otimes M \to M[2-n],$$

satisfying the compatibility conditions which can be written in exactly the same form as compatibility conditions for the higher products m_n^A. All those conditions can be derived from just one property that the cofree $T_+(A[1])$-comodule $T_+(A[1], M) = \oplus_{n \geq 0} A[1]^{\otimes n} \otimes M$ carries a derivation $m^M = (m_n^M)_{n \geq 0}$ such that $[m^M, m^M] = 0$. In particular (M, m_1^M) is a complex of vector spaces.

DEFINITION 5.1.5. Let A be a weakly unital A_∞-algebra. An A-module M is called weakly unital if the cohomology $H^\bullet(M, m_1^M)$ is a unital $H^\bullet(A)$-module.

It is easy to see that left A_∞-modules over A form a dg-category $A - mod$ with morphisms being homomorphisms of the corresponding comodules. As a graded vector space

$$Hom_{A-mod}(M, N) = \oplus_{n \geq 0} \underline{\mathrm{Hom}}_{Vect_k^{\mathbf{Z}}}(A[1]^{\otimes n} \otimes M, N).$$

It easy to see that $Hom_{A-mod}(M, N)$ is a complex.

If M is a right A-module and N is a left A-module then one has a naturally defined structure of a complex on $M \otimes_A N := \oplus_{n \geq 0} M \otimes A[1]^{\otimes n} \otimes N$. The differential is given by the formula:

$$d(x \otimes a_1 \otimes ... \otimes a_n \otimes y) = \sum \pm m_i^M(x \otimes a_1 \otimes ... \otimes a_i) \otimes a_{i+1} \otimes ... \otimes a_n \otimes y) +$$
$$\sum \pm x \otimes a_1 \otimes ... \otimes a_{i-1} \otimes m_k^A(a_i \otimes ... \otimes a_{i+k-1}) \otimes a_{i+k} \otimes ... \otimes a_n \otimes y +$$
$$\sum \pm x \otimes a_1 \otimes ... \otimes a_{i-1} \otimes m_j^N(a_i \otimes ... \otimes a_n \otimes y).$$

We call this complex the *derived tensor product* of M and N.

For any A_∞-algebras A and B we define an $A - B$-bimodule as a $\mathbf{Z}$-graded vector space M together with linear maps

$$c_{n_1, n_2}^M : A[1]^{\otimes n_1} \otimes M \otimes B[1]^{\otimes n_2} \to M[1]$$

satisfying the natural compatibility conditions. If X and Y are formal pointed dg-manifolds corresponding to A and B respectively then an $A - B$-bimodule is the same as a dg-bundle $\mathcal{E}$ over $X \otimes Y$ equipped with a homological vector field $d_{\mathcal{E}}$ which is a lift of the vector field $d_X \otimes 1 + 1 \otimes d_Y$.

EXAMPLE 5.1.6. Let $A = B = M$. We define a structure of diagonal bimodule on A by setting $c_{n_1, n_2}^A = m_{n_1 + n_2 + 1}^A$.

PROPOSITION 5.1.7. *1) To have a structure of an A_∞-module on the complex M is the same as to have a homomorphism of A_∞-algebras $\phi : A \to \underline{End}_{\mathbf{K}}(M)$, where $\mathbf{K}$ is a category of complexes of k-vector spaces.*

2) To have a structure of an $A-B$-bimodule on a graded vector space M is the same as to have a structure of left A-module on M and to have a morphism of A_∞-algebras $\varphi_{A,B} : B^{op} \to Hom_{A-mod}(M, M)$.

Let A be an A_∞-algebra, M be an A-module and $\varphi_{A,A} : A^{op} \to Hom_{A-mod}(M, M)$ be the corresponding morphism of A_∞-algebras. Then the dg-algebra $Centr(\varphi)$ is isomorphic to the dg-algebra $Hom_{A-mod}(M, M)$.

If $M =_A M_B$ is an $A-B$-bimodule and $N =_B N_C$ is a $B-C$-bimodule then the complex ${}_AM_B \otimes_B {}_BN_C$ carries an $A-C$-bimodule structure. It is called the *tensor product* of M and N.

Let $f : X \to Y$ be a homomorphism of formal pointed dg-manifolds corresponding to a homomorphism of A_∞-algebras $A \to B$. Recall that in Section 4 we constructed the formal neighborhood U_f of f in $Maps(X, Y)$ and the A_∞-algebra $Centr(f)$. On the other hand, we have an $A - mod - B$ bimodule structure on B induced by f. Let us denote this bimodule by $M(f)$. We leave the proof of the following result as an exercise to the reader. It will not be used in the paper.

PROPOSITION 5.1.8. *If B is weakly unital then the dg-algebra $End_{A-mod-B}(M(f))$ is quasi-isomorphic to $Centr(f)$.*

A_∞-bimodules will be used later in the study of homologically smooth A_∞-algebras. In the subsequent paper devoted to A_∞-categories we will explain that bimodules give rise to A_∞-functors between the corresponding categories of modules. Tensor product of bimodules corresponds to the composition of A_∞-functors.

5.2. On the tensor product of A_∞-algebras. The tensor product of two dg-algebras A_1 and A_2 is a dg-algebra. For A_∞-algebras there is no canonical simple formula for the A_∞-structure on $A_1 \otimes_k A_2$ which generalizes the one in the dg-algebras case. Some complicated formulas were proposed in [SU2000]. They are not symmetric with respect to the permutation $(A_1, A_2) \mapsto (A_2, A_1)$. We will give below the definition of the dg-algebra which is quasi-isomorphic to the one from [SU2000] in the case when both A_1 and A_2 are weakly unital. Namely, we define the A_∞-tensor product

$$A_1 \text{``} \otimes \text{''} A_2 = End_{A_1-mod-A_2}(A_1 \otimes A_2).$$

Notice that it is a unital dg-algebra. One can show that the dg-category $A-mod-B$ is equivalent (as a dg-category) to $A_1 \text{``} \otimes \text{''} A_2^{op} - mod$.

6. Elliptic spaces

There are interesting examples of A_∞-algebras (in fact dg-algebras) and modules over them coming from geometry of elliptic spaces. Elliptic spaces (see below) fom a dg-category. Although A_∞-categories is a subject of the second volume of the book, the notion of elliptic space is independent of the general theory, so we decided to discuss it here.

6.1. Definition of an elliptic space. Let X be a connected smooth manifold, $W \to X$ be a bundle of finite-dimensional $\mathbf{Z}$-graded complex algebras with the unit. Let $A = \Gamma(X, W)$. Assume that we are given a $\mathbf{C}$-linear map $D : A \to A[1]$, continuous in C^∞-topology, such that $D^2 = 0$, and D satisfies the Leibniz formula. It follows that D is a first order differential operator on W.

DEFINITION 6.1.1. We say that a triple (X, A, D) defines an elliptic space if the corresponding complex is elliptic (equivalently, symbol $\sigma(D)$ defines an acyclic complex at each point of $T^*X \setminus X$).

Let us consider a category $\mathcal{A}$ such that its objects are dg-modules over (A, D) which are projective of finite rank as A-modules.

One can prove the following result.

LEMMA 6.1.2. *Let X be compact, and $N = (X, A, D)$ be an elliptic space. Then for any two objects E, F of $\mathcal{A}$ one has $rk\, H^*(Hom(E, F)) < \infty$.*

Using the lemma one can construct a triangulated category Bun_N (vector bundles over an elliptic space) in the following way. Objects of Bun_N are complex $\mathbf{Z}$-graded vector bundles $E \to X$ such that $\Gamma(X, A \otimes E)$ carries a structure of a dg-algebra over $\Gamma(X, A)$. We define $Hom(E, F)$ to be the space $H^0(\Gamma(X, A \otimes E^* \otimes F)$. It follows from the lemma that this space is finite-dimensional.

REMARK 6.1.3. One can consider vector bundles over an elliptic spaces as objects which are glued from local data, similarly to ordinary vector bundles.

For two elliptic space N_1 and N_2 one can consider functors $F : Bun_{N_1} \to Bun_{N_2}$. They are given by elements from $Bun_{N_1 \times N_2}$.

We are going to give few examples of elliptic spaces. They are of geometric origin. In all the cases $A = \Gamma(X, \bigwedge^*(T_X^* \otimes \mathbf{C}))/I$, where I is a homogeneous ideal, which is invariant under the de Rham differential. We leave as an exercise to the reader to check that in all the examples one gets an elliptic space.

6.2. de Rham complex. Let us take $I = 0$. Then $A = \Omega(X)$. This is the dg-algebra of de Rham differential forms. The whole category of dg-modules is difficult to describe. Some objects are of the form $\Omega(X) \otimes \Gamma(X, E)$, where E is a vector bundle. Then E carries a flat connection.

This example admits a generalization.

DEFINITION 6.2.1. Let $E = \oplus_{i \in \mathbf{Z}} E^i$ be a $\mathbf{Z}$-graded vector bundle over X. A superconnection on E is given by a linear map $\nabla : \Gamma(X, E) \to \Gamma(X, E \otimes T_X^*)$, which satisfies the graded Leibniz identity $\nabla(as) = a\nabla(s) + (-1)^{|a||s|} D(a)s$.

One defines the curvature $curv(\nabla) \in \Gamma(X, End(\bigwedge_X^* \otimes E))^2$ in the natural way (here the superscript denotes the grading).

Then a graded vector bundle E, equipped with a superconnection ∇ such that $curv(\nabla) = 0$ defines a dg-module over $\Omega(X)$.

REMARK 6.2.2. Although the category of vector bundles with flat connections is equivalent to the category of modules over the group algebra of $\pi_1(X)$ (the fundamental group of X), it is not true that the category of projective dg-modules over $\Omega(X)$ is equivalent to the derived category $D^b(\mathbf{C}[\pi_1(X)] - mod)$. In fact $Hom(M, N) \simeq H^*(\underline{Hom}(M, N))$ in the former category, and

$Hom(M, N) \simeq H^*(\pi_1(X), \underline{Hom}(M, N))$ in the latter category. One can guess, that the categories are equivalent for $K(\pi, 1)$-spaces.

6.3. Dolbeault complex. Assume that X admits a complex structure. Then we have: $T_X \otimes \mathbf{C} = T_X^{1,0} \oplus T_X^{0,1}$. Let I be an ideal generated by $(T_X^{1,0})^*$. Then $A = \Omega^{0,*}(X)$ is the dg-algebra of Dolbeault differential forms.

6.4. Foliations with transversal complex structures. Assume that X carries a foliation F which has a transversal complex structure. This means that for any $x \in X$, sufficiently small $U \subset X$ which contains x, the space U/F has a complex structure, and it is compatible with the restriction to an open subset. Then one has a decomposition $T_X/F \otimes \mathbf{C} = T^{0,1} \oplus T^{1,0}$. It follows that $(T^{1,0})^* \subset (T_X/F)^* \otimes \mathbf{C} \subset T_X^* \otimes \mathbf{C}$. Then the ideal I is generated by $(T^{1,0})^*$.

Let Y be a transversal complex manifold (if it exists). Then $A = \Omega^{0,*}(Y) \otimes \Omega^*(F)$, where $\Omega^*(F)$ is a dg-algebra of differential forms along the leaves of F. This example is a combination of the previous ones (de Rham and Dolbeault).

6.5. Lie groups. Let G be a Lie group, H be a Lie subgroup such that G/H has a G-invariant complex structure. For example, one can take $G = SL(n, \mathbf{R})$, $G/H = \mathbf{CP}^{n-1} \setminus \mathbf{RP}^{n-1}$. Let $K_H \subset H$ be a compact subgroup, $\Gamma \subset G$ be a cocompact discrete subgroup. It is clear that $G/K_H \to G/H$ is a bundle with transversal complex structure. The elliptic space we are interested in is given by $\Gamma \backslash G/K_H$. There is a natural projection of this space to a (non-Hausdorff) topological space $\Gamma \backslash G/H$. This is an example of the foliation with a transversal complex structure (it does not exist globally).

6.6. Conformal manifolds. Let X be an oriented $2n$-dimensional manifolds which carries a conformal structure. One introduces the Hodge operator $*$ acting on smooth differential forms, so that $*^2 = (-1)^{2n} = 1$. Let $\Omega^{n,+}(X) = \{\alpha \in \Omega^n(X) | *\alpha = \alpha\}$. We define the graded ideal I such as follows: $I = \Omega^{n,+}(X) \oplus \Omega^{n+1}(X) \oplus \Omega^{n+2}(X) \oplus ... \oplus \Omega^{2n}(X)$.

In particular, taking $n = 2$ one gets autodual connections on a 4-dimensional manifold with conformal structure. Then they give rise to dg-modules over the corresponding dg-algebra $A = \Gamma(X, \bigwedge^*(T_X^* \otimes \mathbf{C}))/I)$.

6.7. 7-dimensional manifolds. Let X be a 7-dimensional smooth manifold. Assume that there exists $\omega \in \Omega^3(X)$ such that $d\omega = 0$.

It is known that the natural action of $GL(7, \mathbf{R})$ in $\bigwedge^3(\mathbf{R^7})$ has an open orbit, and the real points of the stabilizer of each point is isomorphic to the compact form of the exceptional group G_2. More precisely, if S is the stabilizer then $S \simeq G_2 \times \mu_3$, thus $S(\mathbf{R}) \simeq \mathbf{G_2}(\mathbf{R})$. This orbit is also characterized by the property that for any non-zero $v \in \mathbf{R^7}$ the 2-form $\omega(v, x, y)$ is a symplectic form in x, y. Using the fact that $S(\mathbf{R}) \simeq \mathbf{G_2}(\mathbf{R})$ one can prove that the structure group of the tangent bundle to X can be reduced from $GL(7, \mathbf{R})$ to G_2 (i.e. X has an exceptional holonomy G_2). Since G_2 is compact, X carries a Riemannian metric. The ideal I is defined such as follows. It contains all $\bigwedge^m T_X^*$ such that $m \geq 4$. If $m = 2$ one has a decomposition of $\bigwedge^2 T_X^*$ into the sum $V_7 \oplus V_{14}$ of the 7-dimensional and 14-dimensional representations of G_2. We require that V_{14} belongs to I. Similarly, $\bigwedge^3 T_X^*$ is a sum $V_1 \oplus W$, where V_1 is a 1-dimensional representation of G_2 generated by ω, and W is the orthogonal representation. We require that $W \subset I$. Using the fact that ω is a closed form, one can prove that I is invariant under the de Rham differential.

6.8. Hyperkahler manifolds. Let X be a hyperkahler manifold, $\omega_I, \omega_J, \omega_K$ be the corresponding symplectic forms. Then I is generated by these forms.

6.9. Calabi-Yau manifolds. Let X be a Calabi-Yau manifold, $\omega \in \Omega^{1,1}(X)$ be the Kähler form, $\Omega_{vol} \in \Omega^{n,0}(X)$ be a non-degenerate holomorphic form. Then I is generated by $\omega, \Omega_{vol}, \Omega^{0,m}(X), m \geq 2$.

7. Yoneda lemma

7.1. Explicit formulas for the product and differential on $Centr(f)$. Let A be an A_∞-algebra, and $B = End_{\mathbf{K}}(A)$ be the dg-algebra of endomorphisms of A in the category $\mathbf{K}$ of complexes of k-vector spaces. Let $f = f_A : A \to B$ be the natural A_∞-morphism coming from the left action of A on itself. Notice that B is always a unital dg-algebra, while A can be non-unital. The aim of this Section is to discuss the relationship between A and $Centr(f_A)$. This is a simplest case of the A_∞-version of Yoneda lemma.

As a graded vector space $Centr(f_A)$ is isomorphic to $\prod_{n\geq 0} \underline{\mathrm{Hom}}(A^{\otimes(n+1)}, A)[-n]$.

Let us describe the product in $Centr(f)$ for $f = f_A$. Let ϕ, ψ be two homogeneous elements of $Centr(f)$. Then

$$(\phi \cdot \psi)(a_1, a_2, \dots, a_N) = \pm \phi(a_1, \dots, a_{p-1}, \psi(a_p, \dots, a_N)).$$

Here ψ acts on the last group of variables $a_p, \dots, a_N$, and we use the Koszul sign convention for A_∞-algebras in order to determine the sign.

Similarly one has the following formula for the differential (see Section 3.4):

$$(d\phi)(a_1, \dots, a_N) = \sum \pm \phi(a_1, \dots, a_s, m_i(a_{s+1}, \dots, a_{s+i}), a_{s+i+1} \dots, a_N) +$$
$$\sum \pm m_i(a_1, \dots, a_{s-1}, \phi(a_s, \dots, a_j, ..., a_N)).$$

7.2. Yoneda homomorphism. If M is an $A-B$-bimodule then one has a homomorphism of A_∞-algebras $B^{op} \to Centr(\phi_{A,M})$ (see Section 5). We would like to apply this general observation in the case of the diagonal bimodule structure on A. Explicitly, we have the A_∞-morphism $A^{op} \to End_{mod-A}(A)$ or, equivalently, the collection of maps $A^{\otimes m} \to Hom(A^{\otimes n}, A)$. By conjugation it gives us a collection of maps

$$A^{\otimes m} \otimes Hom(A^{\otimes n}, A) \to Hom(A^{\otimes(m+n)}, A).$$

In this way we get a natural A_∞-morphism $Yo : A^{op} \to Centr(f_A)$ called the *Yoneda homomorphism.*

PROPOSITION 7.2.1. *The A_∞-algebra A is weakly unital if and only if the Yoneda homomorphism is a quasi-isomorphism.*

Proof. Since $Centr(f_A)$ is weakly unital, then A must be weakly unital as long as Yoneda morphism is a quasi-isomorphism.

Let us prove the opposite statement. We assume that A is weakly unital. It suffices to prove that the cone $Cone(Yo)$ of the Yoneda homomorphism has trivial cohomology. Thus we need to prove that the cone of the morphism of complexes

$$(A^{op}, m_1) \to (\oplus_{n\geq 1} Hom(A^{\otimes n}, A), m_1^{Centr(f_A)}).$$

is contractible. In order to see this, one considers the extended complex $A \oplus Centr(f_A)$. It has natural filtration arising from the tensor powers of A. The corresponding spectral sequence collapses, which gives an explicit homotopy of the extended complex to the trivial one. This implies the desired quasi-isomorphism of $H^0(A^{op})$ and $H^0(Centr(f_A))$. ∎

REMARK 7.2.2. It look like the construction of $Centr(f_A)$ is the first known canonical construction of a unital dg-algebra quasi-isomorphic to a given A_∞-algebra (canonical but not functorial). This is true even in the case of strictly unital A_∞-algebras. Standard construction via bar and cobar resolutions gives a non-unital dg-algebra.

8. Hochschild cochain and chain complexes of an A_∞-algebra

8.1. Hochschild cochain complex. We change the notation for the homological vector field to Q, since the letter d will be used for the differential. Let $((X,pt),Q)$ be a non-commutative formal pointed dg-manifold corresponding to a non-unital A_∞-algebra A, and $Vect(X)$ the graded Lie algebra of vector fields on X (i.e. continuous derivations of $\mathcal{O}(X)$).

We denote by $C^\bullet(A,A) := C^\bullet(X,X) := Vect(X)[-1]$ the *Hochschild cochain complex* of A. As a $\mathbf{Z}$-graded vector space

$$C^\bullet(A,A) = \prod_{n\geq 0} \underline{\mathrm{Hom}}_{\mathcal{C}}(A[1]^{\otimes n}, A).$$

The differential on $C^\bullet(A,A)$ is given by $[Q,\bullet]$. Algebraically, $C^\bullet(A,A)[1]$ is a DGLA of derivations of the coalgebra $T(A[1])$ (see Section 3).

THEOREM 8.1.1. *Let X be a non-commutative formal pointed dg-manifold and $C^\bullet(X,X)$ be the Hochschild cochain complex. Then one has the following quasi-isomorphism of complexes*

$$C^\bullet(X,X)[1] \simeq T_{id_X}(Maps(X,X)),$$

where T_{id_X} denotes the tangent complex at the identity map.

Proof. Notice that $Maps(Spec(k[\varepsilon]/(\varepsilon^2)) \otimes X, X)$ is the non-commutative dg ind-manifold of vector fields on X. The tangent space T_{id_X} from the theorem can be identified with the set of such $f \in Maps(Spec(k[\varepsilon]/(\varepsilon^2)) \otimes X, X)$ that $f|_{\{pt\}\otimes X} = id_X$. On the other hand the DGLA $C^\bullet(X,X)[1]$ is the DGLA of vector fields on X. The theorem follows. ∎

The Hochschild complex admits a couple of other interpretations. We leave to the reader to check the equivalence of all of them. First, $C^\bullet(A,A) \simeq Centr(id_A)$. Finally, for a weakly unital A one has $C^\bullet(A,A) \simeq Hom_{A-mod-A}(A,A)$. Both are quasi-isomorphisms of complexes.

REMARK 8.1.2. Interpretation of $C^\bullet(A,A)[1]$ as vector fields gives a DGLA structure on this space. It is a Lie algebra of the "commutative" formal group in $Vect_k^{\mathbf{Z}}$, which is an abelianization of the non-commutative formal group of inner (in the sense of tensor categories) automorphisms $\underline{Aut}(X) \subset Maps(X,X)$. Because of this non-commutative structure underlying the Hochschild cochain complex, it is natural to expect that $C^\bullet(A,A)[1]$ carries more structures than just DGLA. Indeed, Deligne's conjecture claims that the DGLA algebra structure on $C^\bullet(A,A)[1]$ can be extended to a structure of an algebra over the operad of singular chains of the topological operad of little discs. Graded Lie algebra structure can be recovered from cells of highest dimension in the cell decomposition of the topological operad.

8.2. Hochschild chain complex. In this subsection we are going to construct a complex of k-vector spaces which is dual to the Hochschild chain complex of a non-unital A_∞-algebra.

8.2.1. *Cyclic differential forms of order zero.* Let (X, pt) be a non-commutative formal pointed manifold over k, and $\mathcal{O}(X)$ the algebra of functions on X. For simplicity we will assume that X is finite-dimensional, i.e. $dim_k\, T_{pt}X < \infty$. If $B = B_X$ is a counital coalgebra corresponding to X (coalgebra of distributions on X) then $\mathcal{O}(X) \simeq B^*$. Let us choose affine coordinates $x_1, x_2, ..., x_n$ at the marked point pt. Then we have an isomorphism of $\mathcal{O}(X)$ with the topological algebra $k\langle\langle x_1, ..., x_n\rangle\rangle$ of formal series in free graded variables $x_1, ..., x_n$.

We define the space of *cyclic differential degree zero forms on* X as

$$\Omega^0_{cycl}(X) = \mathcal{O}(X)/[\mathcal{O}(X), \mathcal{O}(X)]_{top},$$

where $[\mathcal{O}(X), \mathcal{O}(X)]_{top}$ denotes the topological commutator (the closure of the algebraic commutator in the adic topology of the space of non-commutative formal power series).

Equivalently, we can start with the graded k-vector space $\Omega^0_{cycl,dual}(X)$ defined as the kernel of the composition $B \to B \otimes B \to \bigwedge^2 B$ (first map is the coproduct $\Delta : B \to B \otimes B$, while the second one is the natural projection to the skew-symmetric tensors). Then $\Omega^0_{cycl}(X) \simeq (\Omega^0_{cycl,dual}(X))^*$ (dual vector space).

8.2.2. *Higher order cyclic differential forms.* We start with the definition of the *odd tangent bundle* $T[1]X$. This is the dg-analog of the total space of the tangent supervector bundle with the changed parity of fibers. It is more convenient to describe this formal manifold in terms of algebras rather than coalgebras. Namely, the algebra of functions $\mathcal{O}(T[1]X)$ is a unital topological algebra isomorphic to the algebra of formal power series $k\langle\langle x_i, dx_i\rangle\rangle, 1 \leq i \leq n$, where $deg\, dx_i = deg\, x_i + 1$ (we do not impose any commutativity relations between generators). More invariant description involves the odd line. Namely, let $t_1 := Spc(B_1)$, where $(B_1)^* = k\langle\langle \xi\rangle\rangle/(\xi^2), deg\, \xi = +1$. Then we define $T[1]X$ as the formal neighborhood in $Maps(t_1, X)$ of the point p which is the composition of pt with the trivial map of t_1 into the point $Spc(k)$.

DEFINITION 8.2.1. a) The graded vector space

$$\mathcal{O}(T[1]X) = \Omega^\bullet(X) = \prod_{m\geq 0} \Omega^m(X)$$

is called the space of de Rham differential forms on X.

b) The graded space

$$\Omega^0_{cycl}(T[1]X) = \prod_{m\geq 0} \Omega^m_{cycl}(X)$$

is called the space of cyclic differential forms on X.

In coordinate description the grading is given by the total number of dx_i. Clearly each space $\Omega^n_{cycl}(X), n \geq 0$ is dual to some vector space $\Omega^n_{cycl,dual}(X)$ equipped with athe discrete topology (since this is true for $\Omega^0(T[1]X)$).

The *de Rham differential on* $\Omega^\bullet(X)$ corresponds to the vector field $\partial/\partial\xi$ (see description which uses the odd line, it is the same variable ξ). Since Ω^0_{cycl} is given by the natural (functorial) construction, the de Rham differential descends to the subspace of cyclic differential forms. We will denote the former by d_{DR} and the latter by d_{cycl}.

The space of *cyclic 1-forms* $\Omega^1_{cycl}(X)$ is a (topological) span of expressions $x_1x_2...x_l\, dx_j, x_i \in \mathcal{O}(X)$. Equivalently, the space of cyclic 1-forms consists of expressions $\sum_{1\le i\le n} f_i(x_1,...,x_n)\, dx_i$ where $f_i \in k\langle\langle x_1,...,x_n\rangle\rangle$.

There is a map $\varphi : \Omega^1_{cycl}(X) \to \mathcal{O}(X)_{red} := \mathcal{O}(X)/k$, which is defined on $\Omega^1(X)$ by the formula $adb \mapsto [a,b]$ (check that the induced map on the cyclic 1-forms is well-defined). This map does not have an analog in the commutative case.

8.2.3. *Non-commutative Cartan calculus.* Let X be a formal graded manifold over a field k. We denote by $g := g_X$ the graded Lie algebra of continuous linear maps $\mathcal{O}(T[1]X) \to \mathcal{O}(T[1]X)$ generated by de Rham differential $d = d_{DR}$ and contraction maps $i_\xi, \xi \in Vect(X)$ which are defined by the formulas $i_\xi(f) = 0, i_\xi(df) = \xi(f)$ for all $f \in \mathcal{O}(T[1]X)$. Let us define the Lie derivative $Lie_\xi = [d, i_\xi]$ (graded commutator). Then one can easily check the usual formulas of the Cartan calculus

$$[d,d] = 0, Lie_\xi = [d, i_\xi], [d, Lie_\xi] = 0,$$

$$[Lie_\xi, i_\eta] = i_{[\xi,\eta]}, [Lie_\xi, Lie_\eta] = Lie_{[\xi,\eta]}, [i_\xi, i_\eta] = 0,$$

for any $\xi, \eta \in Vect(X)$.

By naturality, the graded Lie algebra g_X acts on the space $\Omega^\bullet_{cycl}(X)$ as well as one the dual space $(\Omega^\bullet_{cycl}(X))^*$.

8.2.4. *Differential on the Hochschild chain complex.* Let Q be a homological vector field on (X, pt). Then $A = T_{pt}X[-1]$ is a non-unital A_∞-algebra.

We define the *dual Hochschild chain complex* $(C_\bullet(A,A))^*$ as $\Omega^1_{cycl}(X)[2]$ with the differential Lie_Q. Our terminology is explained by the observation that $\Omega^1_{cycl}(X)[2]$ is dual to the conventional Hochschild chain complex

$$C_\bullet(A,A) = \oplus_{n\ge 0}(A[1])^{\otimes n} \otimes A.$$

Notice that we use the cohomological grading on $C_\bullet(A,A)$, i.e. chains of degree n in conventional (homological) grading have degree $-n$ in our grading. The differential has degree $+1$.

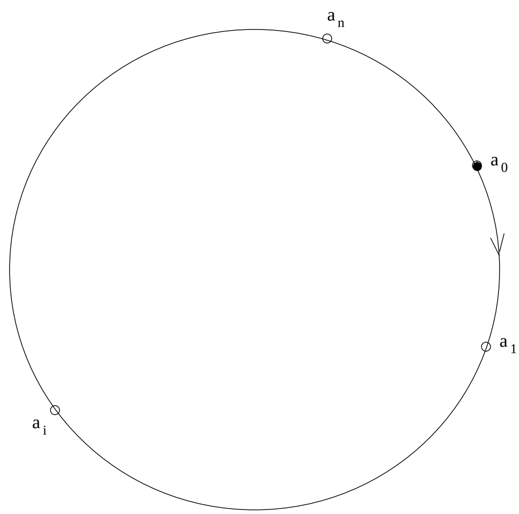
In coordinates the isomorphism identifies an element $f_i(x_1,...,x_n)\otimes x_i \in (A[1]^{\otimes n}\otimes A)^*$ with the homogeneous element $f_i(x_1,...,x_n)\, dx_i \in \Omega^1_{cycl}(X)$. Here $x_i \in (A[1])^*, 1 \le i \le n$ are affine coordinates.

The graded Lie algebra $Vect(X)$ of vector fields of all degrees acts on any functorially defined space, in particular, on all spaces $\Omega^j(X), \Omega^j_{cycl}(X)$, etc. Then we have a differential on $\Omega^j_{cycl}(X)$ given by $b = Lie_Q$ of degree $+1$. There is an explicit formula for the differential b on $C_\bullet(A,A)$ (cf. [T]):

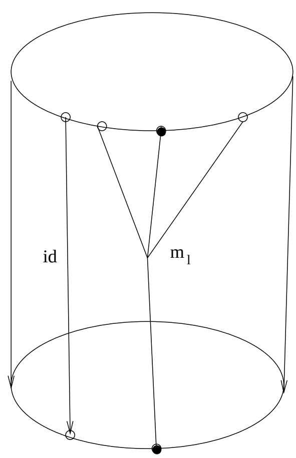
$$b(a_0 \otimes ... \otimes a_n) = \sum \pm a_0 \otimes ... \otimes m_l(a_i \otimes ... \otimes a_j) \otimes ... \otimes a_n$$

$$+ \sum \pm m_l(a_j \otimes ... \otimes a_n \otimes a_0 \otimes ... \otimes a_i) \otimes a_{i+1} \otimes ... \otimes a_{j-1}.$$

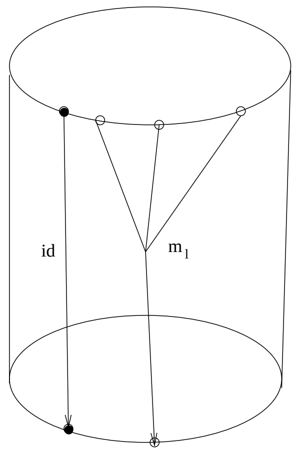
It is convenient to depict a cyclic monomial $a_0 \otimes ... \otimes a_n$ in the following way. We draw a clockwise oriented circle with $n+1$ points labeled from 0 to n such that one point is marked We assign the elements $a_0, a_1, ..., a_n$ to the points with the corresponding labels, putting a_0 at the marked point.



Then we can write $b = b_1 + b_2$ where b_1 is the sum (with appropriate signs) of the expressions depicted below:



Similarly, b_2 is the sum (with appropriate signs) of the expressions depicted below:



In both cases maps m_l are applied to a consequitive cyclically ordered sequence of elements of A assigned to the points on the top circle. The identity map is applied to the remaining elements. Marked point on the top circle is the position of the element of a_0. Marked point on the bottom circle depicts the first tensor factor of the corresponding summand of b. In both cases we start cyclic count of tensor factors clockwise from the marked point.

8.3. The case of strictly unital A_∞-algebras. Let A be a strictly unital A_∞-algebra. There is a *reduced* Hochschild chain complex

$$C_\bullet^{red}(A,A) = \oplus_{n\geq 0} A \otimes ((A/k\cdot 1)[1])^{\otimes n},$$

which is the quotient of $C_\bullet(A,A)$. Similarly there is a reduced Hochschild cochain complex

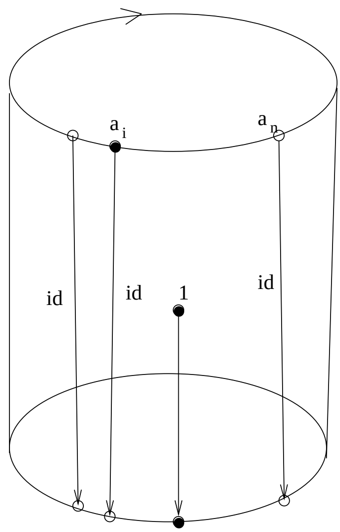
$$C^\bullet_{red}(A,A) = \prod_{n\geq 0} \underline{\mathrm{Hom}}_{\mathcal{C}}((A/k\cdot 1)[1]^{\otimes n}, A),$$

which is a subcomplex of the Hochschild cochain complex $C^\bullet(A,A)$.

Also, $C_\bullet(A,A)$ carries also the "Connes's differential " B of degree -1 (called sometimes "de Rham differential") given by the formula (see [Co94])

$$B(a_0\otimes ...\otimes a_n) = \sum_i \pm 1\otimes a_i\otimes ...\otimes a_n\otimes a_0\otimes ...\otimes a_{i-1}, B^2=0, Bb+bB=0.$$

Here is a graphical description of B (it will receive an explanation in the section devoted to generalized Deligne's conjecture)



Let u be an independent variable of degree $+2$. It follows that for a strictly unital A_∞-algebra A one has a differential $b + uB$ of degree $+1$ on the graded vector space $C_\bullet(A, A)[[u]]$ which makes the latter into a complex called *negative cyclic complex*. In fact $b + uB$ is a differential on a smaller complex $C_\bullet(A, A)[u]$. In the non-unital case one can use Cuntz-Quillen complex instead of a negative cyclic complex (see next subsection).

8.4. Non-unital case: Cuntz-Quillen complex. In this subsection we are going to present a formal dg-version of the mixed complex introduced by Cuntz and Quillen (see [CQ95-1]). In the previous subsection we introduced the Connes differential B in the case of strictly unital A_∞-algebras. In the non-unital case the construction has to be modified. Let $X = A[1]_{form}$ be the corresponding non-commutative formal pointed dg-manifold. The algebra of functions $\mathcal{O}(X) \simeq \prod_{n\geq 0}(A[1]^{\otimes n})^*$ is a complex with the differential Lie_Q.

PROPOSITION 8.4.1. *If A is weakly unital then all non-zero cohomology of the complex $\mathcal{O}(X)$ are trivial, and $H^0(\mathcal{O}(X)) \simeq k$.*

Proof. Let us calculate the cohomology using the spectral sequence associated with the filtration $\prod_{n\geq n_0}(A[1]^{\otimes n})^*$. The term E_1 of the spectral sequence is isomorphic to the complex $\prod_{n\geq 0}((H^\bullet(A[1], m_1))^{\otimes n})^*$ with the differential induced by the multiplication m_2^A on $H^\bullet(A, m_1^A)$. By assumption $H^\bullet(A, m_1^A)$ is a unital algebra, hence all the cohomology groups vanish except of the zeroth one, which is isomorphic to k. This concludes the proof. ■.

It follows from the above Proposition that the complex $\mathcal{O}(X)/k$ is acyclic. We have the following two morphisms of complexes

$$d_{cycl} : (\mathcal{O}(X)/k \cdot 1, Lie_Q) \to (\Omega^1_{cycl}(X), Lie_Q)$$

and

$$\varphi : (\Omega^1_{cycl}(X), Lie_Q) \to (\mathcal{O}(X)/k \cdot 1, Lie_Q).$$

Here d_{cycl} and φ were introduced in the Section 8. We have: $deg(d_{cycl}) = +1, deg(\varphi) = -1, d_{cycl} \circ \varphi = 0, \varphi \circ d_{cycl} = 0$..

Let us consider a *modified* Hochschild chain complex

$$C_\bullet^{mod}(A,A) := (\Omega^1_{cycl}(X)[2])^* \oplus (\mathcal{O}(X)/k\cdot 1)^*$$

with the differential

$$b = \begin{pmatrix} (Lie_Q)^* & \varphi^* \\ 0 & (Lie_Q)^* \end{pmatrix}$$

Let

$B = \begin{pmatrix} 0 & 0 \\ d^*_{cycl} & 0 \end{pmatrix}$ be an endomorphism of $C_\bullet^{mod}(A,A)$ of degree -1. Then $B^2 = 0$. Let u be a formal variable of degree $+2$. We define modified negative cyclic, periodic cyclic and cyclic chain complexes such as follows

$$CC_\bullet^{-,mod}(A) = (C_\bullet^{mod}(A,A)[[u]], b+uB),$$

$$CP_\bullet^{mod}(A) = (C_\bullet^{mod}(A,A)((u)), b+uB),$$

$$CC_\bullet^{mod}(A) = (CP_\bullet^{mod}(A)/CC_\bullet^{-,mod}(A))[-2].$$

For unital dg-algebras these complexes are quasi-isomorphic to the standard ones. If $char\, k = 0$ and A is weakly unital then $CC_\bullet^{-,mod}(A)$ is quasi-isomorphic to the complex $(\Omega^0_{cycl}(X), Lie_Q)^*$. Notice that the $k[[u]]$-module structure on the cohomology $H^\bullet((\Omega^0_{cycl}(X), Lie_Q)^*)$ is not visible from the definition.

9. Homologically smooth and compact A_∞-algebras

From now on we will assume that all A_∞-algebras are weakly unital unless we say otherwise.

9.1. Homological smoothness. Let A be an A_∞-algebra over k and $E_1, E_2, ..., E_n$ be a sequence of A-modules. Let us consider a sequence $(E_{\leq i})_{1\leq i\leq n}$ of A-modules together with exact triangles

$$E_i \to E_{\leq i} \to E_{i+1} \to E_i[1],$$

such that $E_{\leq 1} = E_1$.

We will call $E_{\leq n}$ an *extension* of the sequence $E_1, ..., E_n$.

The reader also notices that the above definition can be given also for the category of $A - A$-bimodules.

Definition 9.1.1. 1) A perfect A-module is the one which is quasi-isomorphic to a direct summand of an extension of a sequence of modules each of which is quasi-isomorphic to $A[n], n \in \mathbf{Z}$.

2) A perfect $A - A$-bimodule is the one which is quasi-isomorphic to a direct summand of an extension of a sequence consisting of bimodules each of which is quasi-isomorphic to $(A \otimes A)[n], n \in \mathbf{Z}$.

Perfect A-modules form a full subcategory $Perf_A$ of the dg-category $A - mod$. Perfect $A - A$-bimodules form a full subcategory $Perf_{A-mod-A}$ of the category of $A - A$-bimodules.[1]

[1]Sometimes $Perf_A$ is called a thick triangulated subcategory of $A - mod$ generated by A. Then it is denoted by $\langle A\rangle$. In the case of $A - A$-bimodules we have a thik triangulated subcategory generated by $A \otimes A$, which is denoted by $\langle A \otimes A\rangle$.

DEFINITION 9.1.2. We say that an A_∞-algebra A is homologically smooth if it is a perfect $A-A$-bimodule (equivalently, A is a perfect module over the A_∞-algebra $A``\otimes" A^{op}$).

REMARK 9.1.3. An $A-B$-bimodule M gives rise to a dg-functor $B-mod \to A-mod$ such that $V \mapsto M \otimes_B V$. The diagonal bimodule A corresponds to the identity functor $Id_{A-mod} : A-mod \to A-mod$. The notion of homological smoothness can be generalized to the framework of A_∞-categories. The corresponding notion of saturated A_∞-category can be spelled out entirely in terms of the identity functor.

Let us list few examples of homologically smooth A_∞-algebras.

EXAMPLE 9.1.4. a) Algebra of functions on a smooth affine scheme.

b) $A = k[x_1, ..., x_n]_q$, which is the algebra of polynomials in variables $x_i, 1 \le i \le n$ subject to the relations $x_i x_j = q_{ij}\, x_j x_i$, where $q_{ij} \in k^*$ satisfy the properties $q_{ii} = 1, q_{ij}q_{ji} = 1$. More generally, all quadratic Koszul algebras, which are deformations of polynomial algebras are homologically smooth.

c) Algebras of regular functions on quantum groups (see [KorSo98]).

d) Free algebras $k\langle x_1, ..., x_n\rangle$.

e) Finite-dimensional associative algebras of finite homological dimension.

f) If X is a smooth scheme over k then the bounded derived category $D^b(Perf(X))$ of the category of perfect complexes (it is equivalent to $D^b(Coh(X))$) has a generator P (see [BvB02]). Then the dg-algebra $A = End(P)$ (here we understand endomorphisms in the "derived sense", see [Ke06]) is a homologically smooth algebra.

Let us introduce an $A-A$-bimodule $A^! = Hom_{A-mod-A}(A, A \otimes A)$ (cf. [Gi2000]). The structure of an $A-A$-bimodule is defined similarly to the case of associative algebras.

PROPOSITION 9.1.5. *If A is homologically smooth then $A^!$ is a perfect $A-A$-bimodule.*

Proof. We observe that $Hom_{C-mod}(C, C)$ is a dg-algebra for any A_∞-algebra C. The Yoneda embedding $C \to Hom_{C-mod}(C, C)$ is a quasi-isomorphism of A_∞-algebras. Let us apply this observation to $C = A \otimes A^{op}$. Then using the A_∞-algebra $A``\otimes" A^{op}$ (see Section 5.2) we obtain a quasi-isomorphism of $A-A$-bimodules $Hom_{A-mod-A}(A \otimes A, A \otimes A) \simeq A \otimes A$. By assumption A is quasi-isomorphic (as an A_∞-bimodule) to a direct summand in an extension of a sequence $(A \otimes A)[n_i]$ for $n_i \in \mathbf{Z}$. Hence $Hom_{A-mod-A}(A \otimes A, A \otimes A)$ is quasi-isomorphic to a direct summand in an extension of a sequence $(A \otimes A)[m_i]$ for $m_i \in \mathbf{Z}$. The result follows. ∎

DEFINITION 9.1.6. The bimodule $A^!$ is called the inverse dualizing bimodule.

The terminology is explained by an observation that if $A = End(P)$ where P is a generator of of $Perf(X)$ (see example 9.1.4f)) then the bimodule $A^!$ corresponds to the functor $F \mapsto F \otimes K_X^{-1}[-dim\, X]$, where K_X is the canonical class of X.[2]

[2]The inverse dualizing module was first mentioned in the paper by M. van den Bergh "Existence theorems for dualizing complexes over non-commutative graded and filtered rings", J. Algebra, 195:2, 1997, 662-679.

REMARK 9.1.7. In [ToVa05] the authors introduced a stronger notion of fibrant dg-algebra. Informally it corresponds to "non-commutative homologically smooth affine schemes of finite type". In the compact case (see the next section) both notions are equivalent.

9.2. Compact A_∞-algebras.

DEFINITION 9.2.1. We say that an A_∞-algebra A is compact if the cohomology $H^\bullet(A, m_1)$ is finite-dimensional.

EXAMPLE 9.2.2. a) If $dim_k A < \infty$ then A is compact.

b) Let X/k be a proper scheme of finite type. According to [BvB02] there exists a compact dg-algebra A such that $Perf_A$ is equivalent to $D^b(Coh(X))$.

c) If $Y \subset X$ is a proper subscheme (possibly singular) of a smooth scheme X then the bounded derived category $D^b_Y(Perf(X))$ of the category of perfect complexes on X, which are supported on Y has a generator P such that $A = End(P)$ is compact. In general it is not homologically smooth for $Y \neq X$. More generally, one can replace X by a formal smooth scheme containing Y, e.g. by the formal neighborhood of Y in the ambient smooth scheme. In particular, for $Y = \{pt\} \subset X = \mathbf{A}^1$ and the generator $\mathcal{O}_Y$ of $D^b(Perf(X))$ the corresponding graded algebra is isomorphic to $k\langle\xi\rangle/(\xi^2)$, where $deg\,\xi = 1$.

PROPOSITION 9.2.3. *If A is compact and homologically smooth then the Hochschild homology and cohomology of A are finite-dimensional.*

Proof. a) Let us start with Hochschild cohomology. We have an isomorphism of complexes $C^\bullet(A, A) \simeq Hom_{A-mod-A}(A, A)$. Since A is homologically smooth the latter complex is quasi-isomorphic to a direct summand of an extension of the bimodule $Hom_{A-mod-A}(A \otimes A, A \otimes A)$. The latter complex is quasi-isomorphic to $A \otimes A$ (see the proof of the Proposition 9.1.5). Since A is compact, the complex $A \otimes A$ has finite-dimensional cohomology. Therefore any perfect $A - A$-bimodule enjoys the same property. We conclude that the Hochschild cohomology groups are finite-dimensional vector spaces.

b) Let us consider the case of Hochschild homology. With any $A-A$-bimodule E we associate a complex of vector spaces $E^\sharp = \oplus_{n \geq 0} A[1]^{\otimes n} \otimes E$ (cf. [Gi2000]). The differential on $E^\sharp$ is given by the same formulas as the Hochschild differential for $C_\bullet(A, A)$ with the only change: we place an element $e \in E$ instead of an element of A at the marked vertex (see Section 8). Taking $E = A$ with the structure of the diagonal $A - A$-bimodule we obtain $A^\sharp = C_\bullet(A, A)$. On the other hand, it is easy to see that the complex $(A \otimes A)^\sharp$ is quasi-isomorphic to (A, m_1), since $(A \otimes A)^\sharp$ is the quotient of the canonical free resolution (bar resolution) for A by a subcomplex A. The construction of $E^\sharp$ is functorial, hence $A^\sharp$ is quasi-isomorphic to a direct summand of an extension (in the category of complexes) of a shift of $(A \otimes A)^\sharp$, because A is smooth. Since $A^\sharp = C_\bullet(A, A)$ we see that the Hochschild homology $H_\bullet(A, A)$ is isomorphic to a direct summand of the cohomology of an extension of a sequence of k-modules $(A[n_i], m_1)$. Since the vector space $H^\bullet(A, m_1)$ is finite-dimensional the result follows. ∎

REMARK 9.2.4. For a homologically smooth compact A_∞-algebra A one has a quasi-isomorphism of complexes $C_\bullet(A, A) \simeq Hom_{A-mod-A}(A^!, A)$ Also, the complex $Hom_{A-mod-A}(M^!, N)$ is quasi-isomorphic to $(M \otimes_A N)^\sharp$ for two $A-A$-bimodules M, N, such that M is perfect. Here $M^! := Hom_{A-mod-A}(M, A \otimes A)$

Having this in mind one can offer a version of the above proof which uses the isomorphism

$$Hom_{A-mod-A}(A^!, A) \simeq Hom_{A-mod-A}(Hom_{A-mod-A}(A, A\otimes A), A).$$

Indeed, since A is homologically smooth the bimodule $Hom_{A-mod-A}(A, A\otimes A)$ is quasi-isomorphic to a direct summand P of an extension of a shift of $Hom_{A-mod-A}(A\otimes A, A\otimes A) \simeq A\otimes A$. Similarly, $Hom_{A-mod-A}(P, A)$ is quasi-isomorphic to a direct summand of an extension of a shift of $Hom_{A-mod-A}(A\otimes A, A\otimes A) \simeq A\otimes A$. Combining the above computations we see that the complex $C_\bullet(A,A)$ is quasi-isomorphic to a direct summand of an extension of a shift of the complex $A\otimes A$. The latter has finite-dimensional cohomology, since A enjoys this property.

Besides algebras of finite quivers there are two main sources of homologically smooth compact $\mathbf{Z}$-graded A_∞-algebras.

EXAMPLE 9.2.5. a) Combining Examples 9.1.4f) and 9.2.2b) we see that the derived category $D^b(Coh(X))$ is equivalent to the category $Perf_A$ for a homologically smooth compact A_∞-algebra A.

b) According to [Se03] the derived category $D^b(F(X))$ of the Fukaya category of a K3 surface X is equivalent to $Perf_A$ for a homologically smooth compact A_∞-algebra A. The latter is generated by Lagrangian spheres, which are vanishing cycles at the critical points for a fibration of X over $\mathbf{CP}^1$. This result can be generalized to other Calabi-Yau manifolds.

In $\mathbf{Z/2}$-graded case examples of homologically smooth compact A_∞-algebras come from Landau-Ginzburg categories (see [Or05], [R03]) and from Fukaya categories for Fano varieties.

REMARK 9.2.6. Formal deformation theory of smooth compact A_∞-algebras gives a finite-dimensional formal pointed (commutative) dg-manifold. The global moduli stack can be constructed using methods of [ToVa05]). It can be thought of as a moduli stack of non-commutative smooth proper varieties.

10. Degeneration Hodge-to-de Rham

10.1. Main conjecture. Let us assume that $char\, k = 0$ and A is a weakly unital A_∞-algebra, which can be $\mathbf{Z}$-graded or $\mathbf{Z/2}$-graded.

For any $n \geq 0$ we define the truncated modified *negative cyclic* complex $C_\bullet^{mod,(n)}(A,A) = C_\bullet^{mod}(A,A)\otimes k[u]/(u^n)$, where $deg\, u = +2$. It is a complex with the differential $b + uB$. Its cohomology will be denoted by $H^\bullet(C_\bullet^{mod,(n)}(A,A))$.

DEFINITION 10.1.1. We say that an A_∞-algebra A satisfies the degeneration property if for any $n \geq 1$ one has: $H^\bullet(C_\bullet^{mod,(n)}(A,A))$ is a flat $k[u]/(u^n)$-module.

CONJECTURE 10.1.2. (Degeneration Hodge-to-de Rham). Let A be a weakly unital compact homologically smooth A_∞-algebra. Then it satisfies the degeneration property.

We will call the above statement the *degeneration conjecture.*

COROLLARY 10.1.3. *If the A satisfies the degeneration property then the negative cyclic homology coincides with $\varprojlim_n H^\bullet(C_\bullet^{mod,(n)}(A,A))$, and it is a flat $k[[u]]$-module.*

REMARK 10.1.4. One can speak about degeneration property (modulo u^n) for A_∞-algebras which are flat over unital commutative k-algebras. For example, let R be an Artinian local k-algebra with the maximal ideal m, and A be a flat R-algebra such that A/m is weakly unital, homologically smooth and compact. Then, assuming the degeneration property for A/m, one can easily see that it holds for A as well. In particular, the Hochschild homology of A gives rise to a vector bundle over $Spec(R) \times \mathbf{A}^1_{form}[-2]$.

Assuming the degeneration property for A we see that there is a $\mathbf{Z}$-graded vector bundle ξ_A over $\mathbf{A}^1_{form}[-2] = Spf(k[[u]])$ with the space of sections isomorphic to

$$\varprojlim_n H^\bullet(C_\bullet^{mod,(n)}(A,A)) = HC_\bullet^{-,mod}(A),$$

which is the negative cyclic homology of A. The fiber of ξ_A at $u=0$ is isomorphic to the Hochschild homology $H_\bullet^{mod}(A,A) := H_\bullet(C_\bullet(A,A))$.

Notice that $\mathbf{Z}$-graded $k((u))$-module $HP_\bullet^{mod}(A)$ of periodic cyclic homology can be described in terms of just one $\mathbf{Z/2}$-graded vector space $HP^{mod}_{even}(A) \oplus \Pi HP^{mod}_{odd}(A)$, where $HP^{mod}_{even}(A)$ (resp. $HP^{mod}_{odd}(A)$) consists of elements of degree zero (resp. degree $+1$) of $HP_\bullet^{mod}(A)$ and Π is the functor of changing the parity. We can interpret ξ_A in terms of ($\mathbf{Z/2}$-graded) supergeometry as a $\mathbf{G}_m$-equivariant supervector bundle over the even formal line $\mathbf{A}^1_{form}$. The structure of a $\mathbf{G}_m$-equivariant supervector bundle ξ_A is equivalent to a filtration F (called Hodge filtration) by even numbers on $HP^{mod}_{even}(A)$ and by odd numbers on $HP^{mod}_{odd}(A)$. The associated $\mathbf{Z}$-graded vector space coincides with $H_\bullet(A,A)$.

We can say few words in support of the degeneration conjecture. One is, of course, the classical Hodge-to-de Rham degeneration theorem (see Section 10.2 below). It is an interesting question to express the classical Hodge theory algebraically, in terms of a generator $\mathcal{E}$ of the derived category of coherent sheaves and the corresponding A_∞-algebra $A = RHom(\mathcal{E},\mathcal{E})$. The degeneration conjecture also trivially holds for algebras of finite quivers without relations.

In classical algebraic geometry there are basically two approaches to the proof of degeneration conjecture. One is analytic and uses Kähler metric, Hodge decomposition, etc. Another one is pure algebraic and uses the technique of reduction to finite characteristic (see [DI87]). Recently Kaledin (see [Kal05]) suggested a proof of a version of the degeneration conjecture based on the reduction to finite characterstic.

Below we will formulate a conjecture which could lead to the definition of crystalline cohomology for A_∞-algebras. Notice that one can define homologically smooth and compact A_∞-algebras over any commutative ring, in particular, over the ring of integers $\mathbf{Z}$. We assume that A is a flat $\mathbf{Z}$-module.

CONJECTURE 10.1.5. Suppose that A is a weakly unital A_∞-algebra over $\mathbf{Z}$, such that it is homologically smooth (but not necessarily compact). Truncated negative cyclic complexes $(C_\bullet(A,A) \otimes \mathbf{Z}[[u,p]]/(u^n,p^m), b+uB)$ and $(C_\bullet(A,A) \otimes \mathbf{Z}[[u,p]]/(u^n,p^m), b-puB)$ are quasi-isomorphic for all $n,m \geq 1$ and all prime numbers p.

If, in addition, A is compact then the homology of either of the above complexes is a flat module over $\mathbf{Z}[[u,p]]/(u^n,p^m)$.

If the above conjecture is true then the degeneration conjecture,probably, can be deduced along the lines of [DI87]. One can also make some conjectures about

Hochschild complex of an arbitrary A_∞-algebra, not assuming that it is compact or homologically smooth. More precisely, let A be a unital A_∞-algebra over the ring of p-adic numbers $\mathbf{Z_p}$. We assume that A is topologically free $\mathbf{Z_p}$-module. Let $A_0 = A \otimes_{\mathbf{Z_p}} \mathbf{Z/p}$ be the reduction modulo p. Then we have the Hochschild complex $(C_\bullet(A_0, A_0), b)$ and the $\mathbf{Z/2}$-graded complex $(C_\bullet(A_0, A_0), b + B)$.

Conjecture 10.1.6. For any i there is natural isomorphism of $\mathbf{Z/2}$-graded vector spaces over the field $\mathbf{Z/p}$:

$$H^\bullet(C_\bullet(A_0, A_0), b) \simeq H^\bullet(C_\bullet(A_0, A_0), b + B).$$

There are similar isomorphisms for weakly unital and non-unital A_∞-algebras, if one replaces $C_\bullet(A_0, A_0)$ by $C_\bullet^{mod}(A_0, A_0)$. Also one has similar isomorphisms for $\mathbf{Z/2}$-graded A_∞-algebras.

The last conjecture presumably gives an isomorphism used in [DI87], but does not imply the degeneration conjecture.

Remark 10.1.7. As we will explain in the second volume there are similar conjectures for saturated A_∞-categories (recall that they are generalizations of homologically smooth compact A_∞-algebras). This observation supports the idea of introducing the category $NCMot$ of non-commutative pure motives. Objects of the latter will be saturated A_∞-categories over a field, and $Hom_{NCMot}(\mathcal{C}_1, \mathcal{C}_2) = K_0(Funct(\mathcal{C}_1, \mathcal{C}_2)) \otimes \mathbf{Q}/equiv$ where K_0 means the K_0-group of the A_∞-category of functors and $equiv$ means numerical equivalence (i.e. the quivalence relation generated by the kernel of the Euler form $\langle E, F\rangle := \chi(RHom(E, F))$, where χ is the Euler characteristic). The above category is worth of consideration and is discussed in [Ko 06]. In particular, one can formulate non-commutative analogs of Weil and Beilinson conjectures for the category $NCMot$.

10.2. Relationship with the classical Hodge theory. Let X be a quasi-projective scheme of finite type over a field k of characteristic zero. Then the category $Perf(X)$ of perfect sheaves on X is equivalent to $H^0(A - mod)$, where $A - mod$ is the category of A_∞-modules over a dg-algebra A. Let us recall a construction of A. Consider a complex E of vector bundles which generates the bounded derived category $D^b(Perf(X))$ (see [BvB]). Then A is quasi-isomorphic to $RHom(E, E)$. More explicitly, let us fix an affine covering $X = \cup_i U_i$. Then the complex $A := \oplus_{i_0, i_1, \dots, i_n} \Gamma(U_{i_0} \cap \dots \cap U_{i_n}, E^* \otimes E)[-n]$, $n = dim\, X$ computes $RHom(E, E)$ and carries a structure of dg-algebra. Different choices of A give rise to equivalent categories $H^0(A - mod)$ (derived Morita equivalence).

Properties of X are encoded in the properties of A. In particular:

a) X is smooth iff A is homologically smooth;

b) X is compact iff A is compact.

Moreover, if X is smooth then

$$H^\bullet(A, A) \simeq Ext^\bullet_{D^b(Coh(X \times X))}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \simeq$$

$$\oplus_{i,j \geq 0} H^i(X, \wedge^j T_X)[-(i + j)]]$$

where $\mathcal{O}_\Delta$ is the structure sheaf of the diagonal $\Delta \subset X \times X$.

Similarly

$$H_\bullet(A, A) \simeq \oplus_{i,j \geq 0} H^i(X, \wedge^j T_X^*)[j - i].$$

The RHS of the last formula is the Hodge cohomology of X. One can consider the hypercohomology $\mathbf{H}^\bullet(X, \Omega_X^\bullet[[u]]/u^n\Omega_X^\bullet[[u]])$ equipped with the differential ud_{dR}. Then the classical Hodge theory ensures degeneration of the corresponding spectral sequence, which means that the hypercohomology is a flat $k[u]/(u^n)$-module for any $n \geq 1$. Usual de Rham cohomology $H_{dR}^\bullet(X)$ is isomorphic to the generic fiber of the corresponding flat vector bundle over the formal line $\mathbf{A}^1_{form}[-2]$, while the fiber at $u = 0$ is isomorphic to the Hodge cohomology $H^\bullet_{Hodge}(X) = \oplus_{i,j\geq 0} H^i(X, \wedge^j T_X^*)[j-i]$. In order to make a connection with the "abstract" theory of the previous subsection we remark that $H_{dR}^\bullet(X)$ is isomorphic to the periodic cyclic homology $HP_\bullet(A)$ while $H_\bullet(A,A)$ is isomorphic to $H^\bullet_{Hodge}(X)$.

11. Symplectic structures and volume forms in non-commutative case

In this section we advocate the following philosophy. Let X be a finite-dimensional non-commutative formal manifold. To define some geometric structure on X means to define a collection of "compatible" such structures on all commutative formal manifolds $\mathcal{M}(X,n) := \widehat{Rep}_0(\mathcal{O}(X), Mat_n(k))$, where $Mat_n(k)$ is the associative algebra of $n \times n$ matrices over k, $\mathcal{O}(X)$ is the algebra of functions on X and $\widehat{Rep}_0(...)$ means the formal completion at the trivial representation. More generally one should consider formal manifolds $\mathcal{M}(X,V) = \widehat{Rep}_0(\mathcal{O}(X), End(V))$, where V is a finite-dimensional object of the tensor category $\mathcal{C}$. For the most of this section we will assume for simplicity that $\mathcal{C} = Vect_k$ or $\mathcal{C} = Vect_k^{\mathbf{Z}}$. We are going to illustrate our approach in two examples: symplectic manifolds and manifolds with volume forms. We would like to say that the compatibility conditions for different n are not clear in the latter case.

11.1. Main definitions. Let (X, pt, Q) be a finite-dimensional formal pointed dg-manifold over a field k of characteristic zero.

Recall that the space of cyclic 1-forms $x_1, \dots, x_n$ can be identified with the direct sum of n copies of the corresponding free algebra A:

$$(a_1, \dots, a_n) \leftrightarrow \sum a_i \otimes dx_i\,.$$

We can define linear operators $\frac{\partial}{\partial x_i} : \mathcal{O}_{cycl}(X) \to \mathcal{O}_{cycl}(X)$ by the formula $dH = \sum_i \frac{\partial H}{\partial x_i} \otimes dx_i$.

We observe that 0-forms are linear combinations of cyclic words (of length ≥ 2) in alphabet $x_1, \dots, x_n$. For example,

$$\frac{\partial(xxyxz)}{\partial x} = xyxz + yxzx + zxxy\,, \quad \frac{\partial(xxyxz)}{\partial y} = xzxx\,,$$

where $xxyxz$ is considered as a cyclic word.

EXERCISE 11.1.1. Check the following identity

$$\sum \left[x_i, \frac{\partial H}{\partial x_i}\right] = 0\,.$$

DEFINITION 11.1.2. A symplectic structure of degree $N \in \mathbf{Z}$ on X is given by a cyclic closed 2-form ω of degree N such that its restriction to the tangent space $T_{pt}X$ is non-degenerate.

In this case the linear map $\xi \mapsto i_\xi \omega$ gives rise to an isomorphism between the space of vector fields on X and the space of cyclic 1-forms.

EXERCISE 11.1.3. Prove that the space of Hamiltonian vector fields (i.e. those preserving ω) is in one-to-one correspondence with the space of cyclic functions (i.e. Hamiltonians of these vector fields).

There is an explicit formula for the Poisson bracket of cyclic functions induced by the symplectic structure:

$$\{G, H\} = \sum \left(\frac{\partial G}{\partial p_i} \otimes \frac{\partial H}{\partial q_i} - \frac{\partial G}{\partial q_i} \otimes \frac{\partial H}{\partial p_i} \right).$$

PROPOSITION 11.1.4. *(Darboux lemma) Symplectic form ω has constant coefficients in some affine coordinates at the point pt. In other words, one can find local coordinates $(x_i)_{i\in I}$ at x_0 such that $\omega = \sum_{i,j\in I} c_{ij} dx_i \otimes dx_j$, where $c_{ij} \in k$.*

Proof. Let us choose an affine structure at the marked point and write down $\omega = \omega_0 + \omega_1 + \omega_2 + ...$, where $\omega_l = \sum_{i,j} c_{ij}(x) dx_i \otimes dx_j$ and $c_{ij}(x)$ is homogeneous of degree l (in particular, ω_0 has constant coefficients). Next we observe that the following lemma holds.

LEMMA 11.1.5. *Let $\omega = \omega_0 + r$, where $r = \omega_l + \omega_{l+1} + ..., l \geq 1$. Then there is a change of affine coordinates $x_i \mapsto x_i + O(x^{l+1})$ which transforms ω into $\omega_0 + \omega_{l+1} + ...$*

Lemma implies the Proposition, since we can make an infinite product of the above changes of variables (it is a well-defined infinite series). The resulting automorphism of the formal neighborhood of x_0 transforms ω into ω_0.

Proof of the lemma. We have $d_{cycl}\omega_j = 0$ for all $j \geq l$. The change of variables is determined by a vector field $v = (v_1, ..., v_n)$ such that $v(x_0) = 0$. Namely, $x_i \mapsto x_i - v_i, 1 \leq i \leq n$. Moreover, we will be looking for a vector field such that $v_i = O(x^{l+1})$ for all i.

We have $Lie_v(\omega) = d(i_v\omega_0) + d(i_v r)$. Since $d\omega_l = 0$ we have $\omega_l = d\alpha_{l+1}$ for some form $\alpha_{l+1} = O(x^{l+1})$ in the obvious notation (formal Poincare lemma). Therefore in order to kill the term with ω_l we need to solve the equation $d\alpha_{l+1} = d(i_v\omega_0)$. It suffices to solve the equation $\alpha_{l+1} = i_v\omega_0$. Since ω_0 is non-degenerate, there exists a unique vector field $v = O(x^{l+1})$ solving last equation. This proves the lemma. ∎

There exists a simple description of closed 2-forms.

THEOREM 11.1.6. *Let $A = \mathcal{O}(X)$ be the (free) algebra of functions on X. Then there exists a canonical isomorphism $\Omega^{2,cl}_{cycl}(X) \simeq [A, A]$.*

Proof. First of all, we define a map $t : \Omega^1_{cycl}(X) \to [A, A]$ by the formula $t(a \otimes db) = [a, b]$. It is clear that this map is onto and it vanishes on $d\Omega^0_{cycl}(X)$. Thus for the associative case we obtain the short sequence

$$A \to A^2/[A, A] \to \Omega^1_{cycl}(X) \to [A, A] \to 0$$

exact everywhere, except of the middle term. If we choose local coordinates, then we obtain a grading on all terms of this sequence. Simple dimension count shows that Euler characteristics of all graded components are zero (we know the generating function of $\Omega^1_{cycl}(X)$, because there exists an an isomorphism $\Omega^1_{cycl}(X) \simeq Der(A)$). Thus the sequence above is exact and coincides with the exact sequence

$$0 \to \Omega^0_{cycl}(X) \to \Omega^1_{cycl}(X) \to \Omega^{2,cl}_{cycl}(X) \to 0\,.$$

This concludes the proof. ■

DEFINITION 11.1.7. Let (X, pt, Q, ω) be a non-commutative formal pointed symplectic dg-manifold. A scalar product of degree N on the A_∞-algebra $A = T_{pt}X[-1]$ is given by a choice of affine coordinates at pt such that the ω becomes constant and gives rise to a non-degenerate bilinear form $A \otimes A \to k[-N]$.

REMARK 11.1.8. Notice that since $Lie_Q(\omega) = 0$ there exists a cyclic function $S \in \Omega^0_{cycl}(X)$ such that $i_Q\omega = dS$ and $\{S, S\} = 0$ (here the Poisson bracket corresponds to the symplectic form ω). It follows that the deformation theory of a non-unital A_∞-algebra A with the scalar product is controlled by the DGLA $\Omega^0_{cycl}(X)$ equipped with the differential $\{S, \bullet\}$.

We can restate the above definition in algebraic terms. Let A be a finite-dimensional A_∞-algebra, which carries a non-degenerate symmetric bilinear form $(,)$ of degree N. This means that for any two elements $a, b \in A$ such that $deg(a) + deg(b) = N$ we are given a number $(a, b) \in k$ such that:

1) for any collection of elements $a_1, ..., a_{n+1} \in A$ the expression $(m_n(a_1, ..., a_n), a_{n+1})$ is cyclically symmetric in the graded sense (i.e. it satisfies the Koszul rule of signs with respect to the cyclic permutation of arguments);

2) bilinear form $(\bullet, \bullet)$ is non-degenerate.

In this case we will say that A is an *A_∞-algebra with the scalar product* of degree N.

The Hamiltonian S can be written as

$$S = \sum_{n \geq 1} \frac{(m_n(a_1, ..., a_n), a_{n+1})}{n+1}.$$

This is a cyclic functional on T_{X,x_0}.

REMARK 11.1.9. One can define a k-linear PROP $\mathcal{P}$ such that $\mathcal{P}$-algebras are associative algebras with non-degenerate scalar products. To to this one observes that the scalar product defines a map $A \otimes A \to k$, while the inverse to it defines a map $k \to A \times A$. Let $\mathcal{P}'$ be a dg-resolution of $\mathcal{P}$. It is natural to say that $\mathcal{P}'$-algebras are A_∞-algebras with scalar product. We conjecture that this definition is equivalent to the above one. In particular, the deformation theories defined in these two ways are equivalent.

11.2. Calabi-Yau structure. The above definition requires A to be finite-dimensional. We can relax this condition requesting that A is compact. As a result we will arrive to a homological version of the notion of scalar product. More precisely, assume that A is weakly unital compact A_∞-algebra. Let $CC^{mod}_\bullet(A) = (CC^{mod}_\bullet(A, A)[u^{-1}], b+uB)$ be the cyclic complex of A. Let us choose a cohomology class $[\varphi] \in H^\bullet(CC^{mod}_\bullet(A))^*$ of degree N. Since the complex (A, m_1) is a subcomplex of $C^{mod}_\bullet(A, A) \subset CC^{mod}_\bullet(A)$ we see that $[\varphi]$ defines a linear functional $Tr_{[\varphi]}$: $H^\bullet(A) \to k[-N]$.

DEFINITION 11.2.1. We say that $[\varphi]$ is homologically non-degenerate if the bilinear form of degree N on $H^\bullet(A)$ given by $(a, b) \mapsto Tr_{[\varphi]}(ab)$ is non-degenerate.

Notice that the above bilinear form defines a symmetric scalar product of degree N on $H^\bullet(A)$.

THEOREM 11.2.2. *For a weakly unital compact A_∞-algebra A a homologically non-degenerate cohomology class $[\varphi]$ gives rise to a class of isomorphisms of non-degenerate scalar products on a minimal model of A.*

Proof. Since $char\, k = 0$ the complex $(CC^{mod}_\bullet(A))^*$ is quasi-isomorphic to $(\Omega^0_{cycl}(X)/k, Lie_Q)$.

LEMMA 11.2.3. *Complex $(\Omega^{2,cl}_{cycl}(X), Lie_Q)$ is quasi-isomorphic to the complex $(\Omega^0_{cycl}(X)/k, Lie_Q)$.*

Proof. Notice that as a complex $(\Omega^{2,cl}_{cycl}(X), Lie_Q)$ is isomorphic to the complex $\Omega^1_{cycl}(X)/d_{cycl}\,\Omega^0_{cycl}(X)$. The latter is quasi-isomorphic to $[\mathcal{O}(X), \mathcal{O}(X)]_{top}$ via $a\,db \mapsto [a,b]$ (recall that $[\mathcal{O}(X), \mathcal{O}(X)]_{top}$ denotes the topological closure of the commutator).

By definition $\Omega^0_{cycl}(X) = \mathcal{O}(X)/[\mathcal{O}(X), \mathcal{O}(X)]_{top}$. We know that $\mathcal{O}(X)/k$ is acyclic, hence $\Omega^0_{cycl}(X)/k$ is quasi-isomorphic to $[\mathcal{O}(X), \mathcal{O}(X)]_{top}$. Hence the complex $(\Omega^{2,cl}_{cycl}(X), Lie_Q)$ is quasi-isomorphic to $(\Omega^0_{cycl}(X)/k, Lie_Q)$. ■

As a corollary we obtain an isomorphism of cohomology groups $H^\bullet(\Omega^{2,cl}_{cycl}(X)) \simeq H^\bullet(\Omega^0_{cycl}(X)/k)$. Having a non-degenerate cohomology class $[\varphi] \in H^\bullet(CC^{mod}_\bullet(A))^* \simeq H^\bullet(\Omega^{2,cl}_{cycl}(X), Lie_Q)$ as above, we can choose its representative $\omega \in \Omega^{2,cl}_{cycl}(X)$, $Lie_Q\omega = 0$. Let us consider $\omega(x_0)$. It can be described pure algebraically such as follows. Notice that there is a natural projection $H^\bullet(\Omega^0_{cycl}(X)/k) \to (A/[A,A])^*$ which corresponds to the taking the first Taylor coefficient of the cyclic function. Then the above evaluation $\omega(x_0)$ is the image of $\varphi(x_0)$ under the natural map $(A/[A,A])^* \to (Sym^2(A))^*$ which assigns to a linear functional l the bilinear form $l(ab)$.

We claim that the total map $H^\bullet(\Omega^{2,cl}_{cycl}(X)) \to (Sym^2(A))^*$ is the same as the evaluation at x_0 of the closed cyclic 2-form. Equivalently, we claim that $\omega(x_0)(a,b) = Tr_\varphi(ab)$. Indeed, if $f \in \Omega^0_{cycl}(X)/k$ is the cyclic function corresponding to ω then we can write $f = \sum_i a_i x_i + O(x^2)$. Therefore $Lie_Q(f) = \sum_{l,i,j} a_i c^{ij}_l [x_i, x_j] + O(x^3)$, where c^{ij}_l are structure constants of $\mathcal{O}(X)$. Dualizing we obtain the claim.

PROPOSITION 11.2.4. *Let ω_1 and ω_2 be two symplectic structures on the finite-dimensional formal pointed minimal dg-manifold (X, pt, Q) such that $[\omega_1] = [\omega_2]$ in the cohomology of the complex $(\Omega^{2,cl}_{cycl}(X), Lie_Q)$ consisting of closed cyclic 2-forms. Then there exists a change of coordinates at x_0 preserving Q which transforms ω_1 into ω_2.*

COROLLARY 11.2.5. *Let (X, pt, Q) be a (possibly infinite-dimensional) formal pointed dg-manifold endowed with a (possibly degenerate) closed cyclic 2-form ω. Assume that the tangent cohomology $H^0(T_{pt}X)$ is finite-dimensional and ω induces a non-degenerate pairing on it. Then on the minimal model of (X, pt, Q) we have a canonical isomorphism class of symplectic forms modulo the action of the group $Aut(X, pt, Q)$.*

Proof. Let M be a (finite-dimensional) minimal model of A. Choosing a cohomology class $[\varphi]$ as above we obtain a non-degenerate bilinear form on M, which is the restriction $\omega(x_0)$ of a representative $\omega \in \Omega^{2,cl}(X)$. By construction this scalar product depends on ω. We would like to show that in fact it depends on the cohomology class of ω, i.e. on φ only. This is the corollary of the following result.

LEMMA 11.2.6. *Let $\omega_1 = \omega + Lie_Q(d\alpha)$. Then there exists a vector field v such that $v(x_0) = 0, [v, Q] = 0$ and $Lie_v(\omega) = Lie_Q(d\alpha)$.*

Proof. As in the proof of Darboux lemma we need to find a vector field v, satisfying the condition $di_v(\omega) = Lie_Q(d\alpha)$. Let $\beta = Lie_Q(\alpha)$. Then $d\beta = dLie_Q(\alpha) = 0$. Since ω is non-degenerate we can find v satisfying the conditions of the Proposition and such that $di_v(\omega) = Lie_Q(d\alpha)$. Using this v we can change affine coordinates transforming $\omega + Lie_Q(d\alpha)$ back to ω. This concludes the proof of the Proposition and the Theorem.■

We will sometimes call the cohomology class $[\varphi]$ a *Calabi-Yau structure* on A (or on the corresponding non-commutative formal pointed dg-manifold X). The following example illustrates the relation to geometry.

EXAMPLE 11.2.7. Let X be a complex Calabi-Yau manifold of dimension n. Then it carries a nowhere vanishing holomorphic n-form vol. Let us fix a holomorphic vector bundle E and consider a dg-algebra $A = \Omega^{0,*}(X, End(E))$ of Dolbeault $(0,p)$-forms with values in $End(E)$. This dg-algebra carries a linear functional $a \mapsto \int_X Tr(a) \wedge vol$. One can check that this is a cyclic cocycle which defines a non-degenerate pairing on $H^\bullet(A)$ in the way described above.

There is another approach to Calabi-Yau structures in the case when A is homologically smooth. Namely, we say that A carries a Calabi-Yau structure of dimension N if $A^! \simeq A[N]$ (recall that $A^!$ is the $A-A$-bimodule $Hom_{A-mod-A}(A, A \otimes A)$ introduced in Section 8.1. Then we expect the following conjecture to be true.

CONJECTURE 11.2.8. If A is a homologically smooth compact finite-dimensional A_∞-algebra then the existence of a non-degenerate cohomology class $[\varphi]$ of degree $dim\, A$ is equivalent to the condition $A^! \simeq A[dim\, A]$.

If A is the dg-algebra of endomorphisms of a generator of $D^b(Coh(X))$ (X is Calabi-Yau) then the above conjecture holds trivially.

Finally, we would like to illustrate the relationship of the non-commutative symplectic geometry discussed above with the commutative symplectic geometry of certain spaces of representations. More generally we would like to associate with $X = Spc(T(A[1]))$ a collection of formal algebraic varieties, so that some "non-commutative" geometric structure on X becomes a collection of compatible "commutative" structures on formal manifolds $\mathcal{M}(X,n) := \widehat{Rep_0}(\mathcal{O}(X), Mat_n(k))$, where $Mat_n(k)$ is the associative algebra of $n \times n$ matrices over k, $\mathcal{O}(X)$ is the algebra of functions on X and $\widehat{Rep_0}(...)$ means the formal completion at the trivial representation. In other words, we would like to define a collection of compatible geometric structure on "$Mat_n(k)$-points" of the formal manifold X. In the case of symplectic structure this philosophy is illustrated by the following result.

THEOREM 11.2.9. *Let X be a non-commutative formal symplectic manifold in $Vect_k$. Then it defines a collection of symplectic structures on all manifolds $\mathcal{M}(X,n), n \geq 1$.*

Proof. Let $\mathcal{O}(X) = A, \mathcal{O}(\mathcal{M}(X,n)) = B$. Then we can choose isomorphisms $A \simeq k\langle\langle x_1, ..., x_m\rangle\rangle$ and $B \simeq \langle\langle x_1^{\alpha,\beta}, ..., x_m^{\alpha,\beta}\rangle\rangle$, where $1 \leq \alpha, \beta \leq n$. To any $a \in A$ we can assign $\widehat{a} \in B \otimes Mat_n(k)$ such that:

$$\hat{x}_i = \sum_{\alpha,\beta} x_i^{\alpha,\beta} \otimes e_{\alpha,\beta},$$

where $e_{\alpha,\beta}$ is the $n \times n$ matrix with the only non-trivial element equal to 1 on the intersection of α-th line and β-th column. The above formulas define an algebra homomorphism. Composing it with the map $id_B \otimes Tr_{Mat_n(k)}$ we get a linear map $\mathcal{O}_{cycl}(X) \to \mathcal{O}(\mathcal{M}(X,n))$. Indeed the closure of the commutator $[A,A]$ is mapped to zero. Similarly, we have a morphism of complexes $\Omega^\bullet_{cycl}(X) \to \Omega^\bullet(\mathcal{M}(X,n))$, such that

$$dx_i \mapsto \sum_{\alpha,\beta} dx_i^{\alpha,\beta} e_{\alpha,\beta}.$$

Clearly, continuous derivations of A (i.e. vector fields on X) give rise to the vector fields on $\mathcal{M}(X,n)$.

Finally, one can see that a non-degenerate cyclic 2-form ω is mapped to the tensor product of a non-degenerate 2-form on $\mathcal{M}(X,n)$ and a nondegenerate 2-form $Tr(XY)$ on $Mat_n(k)$. Therefore a symplectic form on X gives rise to a symplectic form on $\mathcal{M}(X,n), n \geq 1$. ∎

11.3. Volume forms. We will assume for simplicity that $k = \mathbf{C}$ and $\mathcal{C} = Vect_{\mathbf{C}}$. Let (X, x_0) be a finite-dimensional non-commutative formal pointed manifold. Choosing local coordinates we fix an isomorphism of topological algebras $A := \mathcal{O}(X) \simeq \widehat{T}(V)$ (completed tensor algebra of a finite-dimensional vector space V).

We would like to define a class of volume forms on X. For each $n \geq 1$ let us choose local coordinates on $\mathcal{M}(X,n)$ as well as a tensor $\rho \in \mathcal{O}_{cycl}(X)\widehat{\otimes}\mathcal{O}_{cycl}(X)$. Then we can formally write $\rho = \sum_m a_m \otimes b_m$ (possibly infinite sum). Then we can define a volume form on $\mathcal{M}(X,n)$ such as follows:

$vol((x_i)_{i\in I}, \rho) = \bigwedge^{top}(dx_i)^{\alpha,\beta} exp(\sum_m Tr(a_m)Tr(b_m))$.

It is easy to see that this is a well-defined volume form.

THEOREM 11.3.1. *Let $(x_i')_{i\in I}$ be another choice of local coordinates. Then there exists $\rho' \in \mathcal{O}_{cycl}(X)\widehat{\otimes}\mathcal{O}_{cycl}(X)$ such that $vol((x_i)_{i\in I}, \rho) = vol((x_i')_{i\in I}, \rho')$.*

Proof. Let us define a linear map $div : Vect(X) \to \mathcal{O}_{cycl}(X)\widehat{\otimes}\mathcal{O}_{cycl}(X)$ in the following way. Let $\xi = \sum_{i,j_1,\dots,j_m} c^i_{j_1,\dots,j_m} x_1^{j_1}...x_m^{j_m}\partial/\partial x_i$. Then it defines a vector field ξ_1 on $\mathcal{M}(X,n)$, which is isomorphic to the formal neighborhood of $0 \in \mathbf{C}^{\mathbf{n^2|I|}}$. The latter space carries a standard volume form Vol, so we have the divergence of the vector field xi_1, defined in the usual way: $Lie_{\xi_1}(Vol) = div(\xi_1)Vol$. We leave to the reader to check that $div(\xi_1) = \sum_{j_1,\dots,j_m,j_p=i} c^i_{j_1,\dots,j_m} Tr(X_{j_1}...X_{j_{p-1}})Tr(X_{j_{p+1}}...X_{j_m})$, where $X_r \in Mat_n(\mathbf{C})$ is the image of the local coordinate x_r. Replacing X_i by $x_i, i \in I$ we obtain the desired element $div(\xi)$.

We see that an infinitesimal cahnge of coordinates by the vector field ξ leads to the multiplication of $vol((x_i)_{i\in I}, \rho)$ by the image of $exp(div(\xi))$. But all the traces go to zero under the commutator map, so the volume form does not change. This concludes the proof. ∎.

We would like to interpret a collection $vol((x_i)_{i\in I}, \rho), n \geq 1$ as a single class of volume forms on X.

11.3.1. *Digression about matrix integrals.* We conclude this section with a remark on matrix integrals. It often appears in quantum field theory or string theory that one needs to compute integrals of over the spaces of $n \times n$ matrices, and then take the limit $n \to +\infty$. A typical integral is of the form

$$I_n = \int_{Herm_n} exp(\sum_m Tr(X_{j_1}...X_{j_m})Tr(X_{j_{m+1}}...X_{j_n}))d^{n^2}X,$$

where we integrate over the space of all $n \times n$ Hermitian matrices, and the expression in the exponent is cyclically invariant. The above theorem suggest to interpret integrals like I_n as volumes of the spaces of matrices with respect to some volume form $vol((x_i)_{i \in I}, \rho)$.

EXAMPLE 11.3.2. For one-matrix model one has $I_n = exp(-nTr(f(X)))d^{n^2}X$, where $f : \mathbf{R} \to \mathbf{R}$ is a function decreasing sufficiently fast at $\pm\infty$. The factor n can be written as $Tr(id)$, so one can interpret I_n as desired. We remark that as $n \to +\infty$ one has

$$log(I_n) = -n^2 log(n)/2 + \sum_{g \geq 0} c_g n^{2-2g}.$$

12. Hochschild complexes as algebras over operads and PROPs

Let A be a strictly unital A_∞-algebra over a field k of characteristic zero. In this section we are going to describe a colored dg-operad P such that the pair $(C^\bullet(A, A), C_\bullet(A, A))$ is an algebra over this operad. More precisely, we are going to describe $\mathbf{Z}$-graded k-vector spaces $A(n, m)$ and $B(n, m)$, $n, m \geq 0$ which are components of the colored operad such that $B(n, m) \neq 0$ for $m = 1$ only and $A(n, m) \neq 0$ for $m = 0$ only together with the colored operad structure and the action

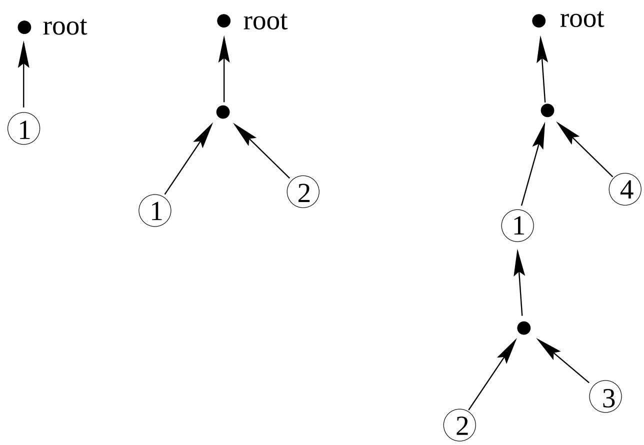
a) $A(n, 0) \otimes (C^\bullet(A, A))^{\otimes n} \to C^\bullet(A, A)$,

b) $B(n, 1) \otimes (C^\bullet(A, A))^{\otimes n} \otimes C_\bullet(A, A) \to C_\bullet(A, A)$.

Then, assuming that A carries a non-degenerate scalar product, we are going to describe a PROP R associated with moduli spaces of Riemannian surfaces and a structure of R-algebra on $C_\bullet(A, A)$.

12.1. Configuration spaces of discs. We start with the spaces $A(n, 0)$. They are chain complexes. The complex $A(n, 0)$ coincides with the complex M_n of the minimal operad $M = (M_n)_{n \geq 0}$ described in [KoSo2000], Section 5. Without going into details which can be found in loc. cit. we recall main facts about the operad M. A basis of M_n as a k-vector space is formed by n-labeled planar trees (such trees have internal vertices labeled by the set $\{1, ..., n\}$ as well as other internal vertices which are non-labeled and each has the valency at least 3).

We can depict n-labeled trees such as follows

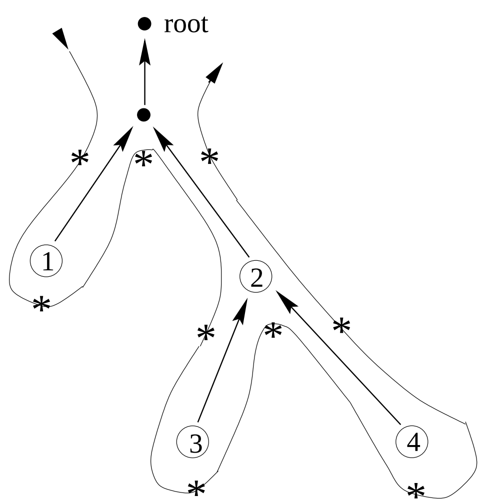


Labeled vertices are depicted as circles with numbers inscribed, non-labeled vertices are depicted as black vertices. In this way we obtain a graded operad M with the total degree of the basis element corresponding to a tree T equal to

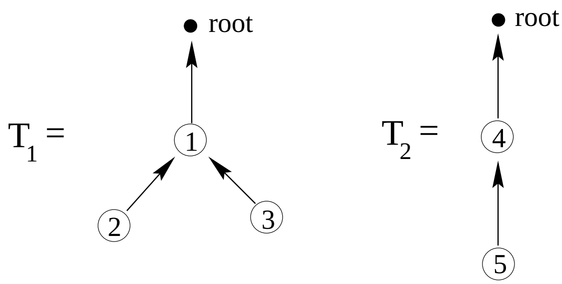
$$deg(T) = \sum_{v \in V_{lab}(T)} (1 - |v|) + \sum_{v \in V_{nonl}(T)} (3 - |v|)$$

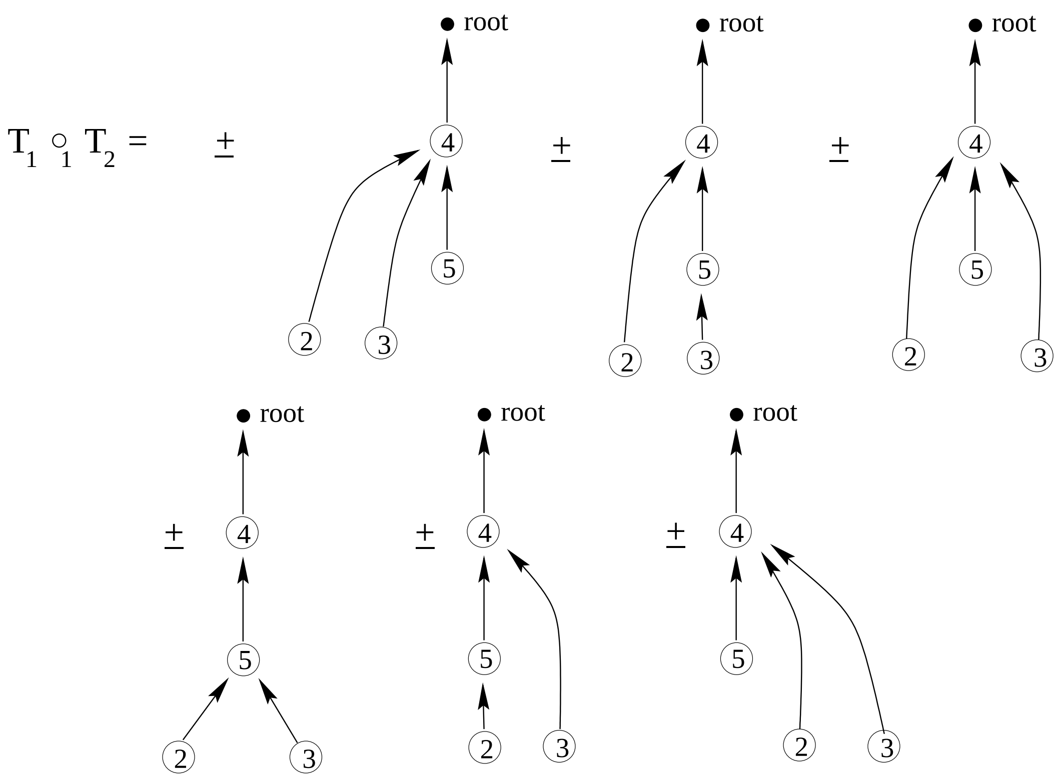
where $V_{lab}(T)$ and $V_{nonl}(T)$ denote the sets of labeled and non-labeled vertices respectively , and $|v|$ is the valency of the vertex v, i.e. the cardinality of the set of edges attached to v.

The notion of an *angle* between two edges incoming in a vertex is illustrated in the following figure (angles are marked by asteriscs).



Operadic composition and the differential are described in [KoSo2000], sections 5.2, 5.3. We borrow from there the following figure which illustrates the operadic composition of generators corresponding to labeled trees T_1 and T_2.





Informally speaking, the operadic gluing of T_2 to T_1 at an internal vertex v of T_1 is obtained by:

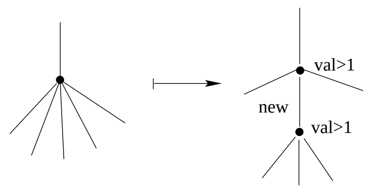
a) Removing from T_1 the vertex v together with all incoming edges and vertices.

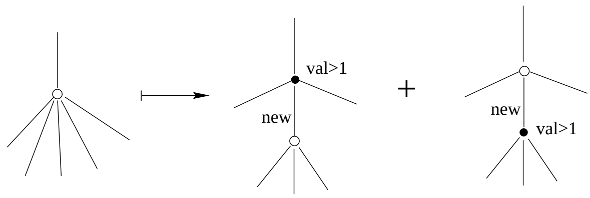
b) Gluing T_2 to v (with the root vertex removed from T_2). Then

c) Inserting removed vertices and edges of T_1 in all angles between incoming edges to the new vertex v_{new}.

d) Taking the sum (with appropriate signs) over all possible inserting of edges in c).

The differential d_M is a sum of the "local" differentials d_v, where v runs through the set of all internal vertices. Each d_v inserts a new edge into the set of edges attached to v. The following figure illustrates the difference between labeled (white) and non-labeled (black) vertices.





In this way we make M into a dg-operad. It was proved in [KoSo1], that M is quasi-isomorphic to the dg-operad $Chains(FM_2)$ of singular chains on the Fulton-Macpherson operad FM_2. The latter consists of the compactified moduli spaces of configurations of points in $\mathbf{R^2}$ (see e.g. [KoSo2000], Section 7.2 for a description). It was also proved in [KoSo2000] (see also Chapter 5) that $C^\bullet(A, A)$ is an algebra over the operad M (Deligne's conjecture follows from this fact). The operad FM_2 is homotopy equivalent to the famous operad $C_2 = (C_2(n))_{n\geq 0}$ of 2-dimensional discs (little disc operad). Thus $C^\bullet(A, A)$ is an algebra (in the homotopy sense) over the operad $Chains(C_2)$.

12.2. Configurations of points on the cylinder. Let $\Sigma = S^1 \times [0, 1]$ denotes the standard cylinder.

Let us denote by $S(n)$ the set of isotopy classes of the following graphs $\Gamma \subset \Sigma$:

a) every graph Γ is a forest (i.e. disjoint union of finitely many trees $\Gamma = \sqcup_i T_i$);

b) the set of vertices $V(\Gamma)$ is decomposed into the union $V_{\partial\Sigma} \sqcup V_{lab} \sqcup V_{nonl} \sqcup V_1$ of four sets with the following properties:

b1) the set $V_{\partial\Sigma}$ is the union $\{in\} \cup \{out\} \cup V_{out}$ of three sets of points which belong to the boundary $\partial\Sigma$ of the cylinder. The set $\{in\}$ consists of one marked point which belongs to the boundary circle $S^1 \times \{1\}$ while the set $\{out\}$ consists of one marked point which belongs to the boundary circle $S^1 \times \{0\}$. The set V_{out} consists of a finitely many unlableled points on the boundary circle $S^1 \times \{0\}$;

b2) the set V_{lab} consists of n labeled points which belong to the surface $S^1 \times (0, 1)$ of the cylinder;

b3) the set V_{nonl} consists of a finitely many non-labeled points which belong to the surface $S^1 \times (0, 1)$ of the cylinder;

b4) the set V_1 is either empty or consists of only one element denoted by $\mathbf{1} \in S^1 \times (0, 1)$ and called *special* vertex;

c) the following conditions on the valencies of vertices are imposed:

c1) the valency of the vertex *out* is less or equal than 1;

c2) the valency of each vertex from the set $V_{\partial\Sigma} \setminus V_{out}$ is equal to 1;

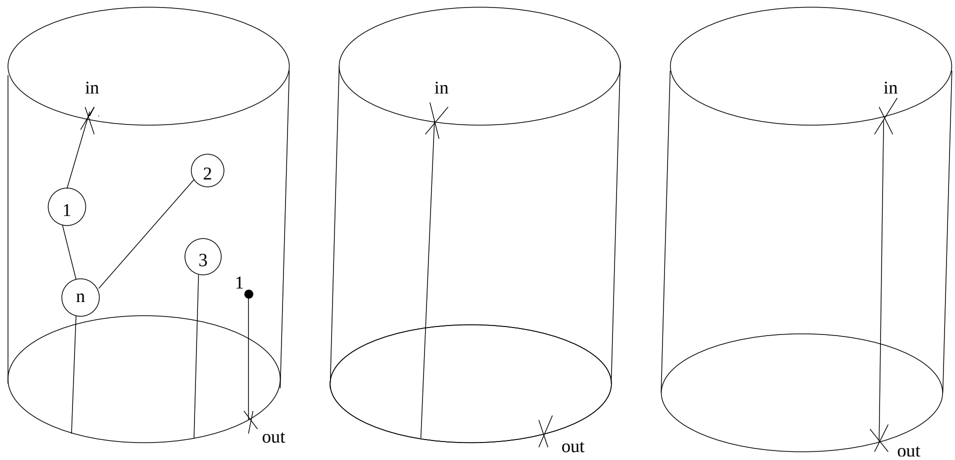
c3) the valency of each vertex from V_{lab} is at least 1;

c4) the valency of each vertex from V_{nonl} is at least 3;

Ic5) if the set V_1 is non-empty then the valency of the special vertex is equal to 1. In this case the only outcoming edge connects $\mathbf{1}$ with the vertex *out*.

d) Every tree T_i from the forest Γ has its root vertex in the set $V_{\partial\Sigma}$.

e) We orient each tree T_i down to its root vertex.



REMARK 12.2.1. Let us consider the configuration space $X_n, n \geq 0$ which consists of (modulo $\mathbf{C}^*$-dilation) equivalence classes of n points on $\mathbf{CP}^1 \setminus \{0, \infty\}$ together with two direction lines at the tangent spaces at the points 0 and ∞. One-point compactification $\widehat{X}_n$ admits a cell decomposition with cells (except of the point $\widehat{X}_n \setminus X_n$) parametrized by elements of the set $S(n)$. This can be proved with the help of Strebel differentials (cf. [KoSo2000], Section 5.5).

Previous remark is related to the following description of the sets $S(n)$ (it will be used later in the paper). Let us contract both circles of the boundary $\partial\Sigma$ into points. In this way we obtain a tree on the sphere. Points become vertices of the tree and lines outcoming from the points become edges. There are two vertices marked by *in* and *out* (placed at the north and south poles respectively). We orient the tree towards to the vertex *out*. An additional structure consists of:

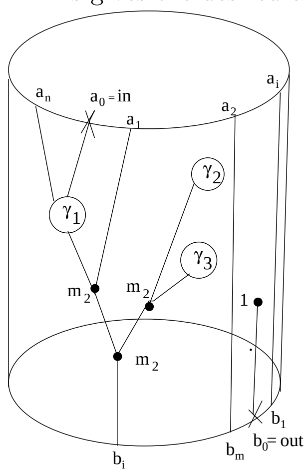
a) Marked edge outcoming from *in* (it corresponds to the edge outcoming from *in*).

b) Either a marked edge incoming to *out* (there was an edge incoming to *out* which connected it with a vertex not marked by $\mathbf{1}$) or an angle between two edges incoming to *out* (all edges which have one of the endpoint vertices on the bottom circle become after contracting it to a point the edges incoming to *out*, and if there was an edge connecting a point marked by $\mathbf{1}$ with *out*, we mark the angle between edges containing this line).

The reader notices that the star of the vertex *out* can be identified with a regular k-gon, where k is the number of incoming to *out* edges. For this k-gon we have either a marked point on an edge (case a) above) or a marked angle with the vertex in *out* (case b) above).

12.3. Generalization of Deligne's conjecture. The definition of the operadic space $B(n,1)$ will be clear from the description of its action on the Hochschild chain complex. The space $B(n,1)$ will have a basis parametrized by elements of the set $S(n)$ described in the previous subsection. Let us describe the action of a generator of $B(n,1)$ on a pair $(\gamma_1 \otimes ... \otimes \gamma_n, \beta)$, where $\gamma_1 \otimes ... \otimes \gamma_n \in C^\bullet(A,A)^{\otimes n}$ and

$\beta = a_0 \otimes a_1 \otimes ... \otimes a_l \in C_l(A, A)$. We attach elements $a_0, a_1, ..., a_l$ to points on Σ_h^{in}, in a cyclic order, such that a_0 is attached to the point *in*. We attach γ_i to the ith numbered point on the surface of Σ_h. Then we draw disjoint continuous segments (in all possible ways, considering pictures up to an isotopy) starting from each point marked by some element a_i and oriented downstairs, with the requirements a)-c) as above, with the only modification that we allow an arbitrary number of points on $S^1 \times \{1\}$. We attach higher multiplications m_j to all non-numbered vertices, so that j is equal to the incoming valency of the vertex. Reading from the top to the bottom and composing γ_i and m_j we obtain (on the bottom circle) an element $b_0 \otimes ... \otimes b_m \in C_\bullet(A, A)$ with b_0 attached to the vertex *out*. If the special vertex **1** is present then we set $b_0 = 1$. This gives the desired action.



Composition of the operations in $B(n, 1)$ corresponds to the gluing of the cylinders such that the point *out* of the top cylinder is identified with the point *in* of the bottom cylinder. If after the gluing there is a line from the point marked **1** on the top cylinder which does not end at the point *out* of the bottom cylinder, we will declare such a composition to be equal to zero.

Let us now consider a topological colored operad $C_2^{col} = (C_2^{col}(n, m))_{n,m\geq 0}$ with two colors such that $C_2^{col}(n, m) \neq \emptyset$ only if $m = 0, 1$, and

a) In the case $m = 0$ it is the little disc operad.

b) In the case $m = 1$ $C_2^{col}(n, 1)$ is the moduli space (modulo rotations) of the configurations of $n \geq 1$ discs on the cyliner $S^1 \times [0, h]$ $h \geq 0$, and two marked points on the boundary of the cylinder. We also add the degenerate circle of configurations $n = 0, h = 0$. The topological space $C_2^{col}(n, 1)$ is homotopically equivalent to the configuration space X_n described in the previous subsection.

Let $Chains(C_2^{col})$ be the colored operad of singular chains on C_2^{col}. Then, similarly to [KoSo2000], Section 7, one proves (using the explicit action of the colored operad $P = (A(n, m), B(n, m))_{n,m\geq 0}$ described above) the following result.

THEOREM 12.3.1. *Let A be a unital A_∞-algebra. Then the pair $(C^\bullet(A, A), C_\bullet(A, A))$ is an algebra over the colored operad $Chains(C_2^{col})$ (which is quasi-isomorphic to P) such that for $h = 0, n = 0$ and coinciding points in $=$ out, the corresponding operation is the identity.*

REMARK 12.3.2. The above Theorem generalizes Deligne's conjecture. It is related to the abstract calculus associated with A (see [TaT2000], [TaT05]). The reader also notices that for $h = 0, n = 0$ we have the moduli space of two points on the circle. It is homeomorphic to S^1. Thus we have an action of S^1 on $C_\bullet(A, A)$. This action gives rise to the Connes differential B.

Similarly to the case of little disc operad, one can prove the following result.

PROPOSITION 12.3.3. *The colored operad* C_2^{col} *is formal, i.e. it is quasi-isomorphic to its homology colored operad.*

If A is non-unital we can consider the direct sum $A_1 = A \oplus k$ and make it into a unital A_∞-algebra. The *reduced* Hochschild chain complex of A_1 is defined as $C_\bullet^{red}(A_1, A_1) = \oplus_{n \geq 0} A_1 \otimes ((A_1/k)[1])^{\otimes n}$ with the same differential as in the unital case. One defines the reduced Hochschild cochain complex $C^\bullet_{red}(A_1, A_1)$ similarly. We define the *modified* Hochschild chain complex $C_\bullet^{mod}(A, A)$ from the following isomorphism of complexes $C_\bullet^{red}(A_1, A_1) \simeq C_\bullet^{mod}(A, A) \oplus k$. Similarly, we define the modified Hochschild cochain complex from the decomposition $C^\bullet_{red}(A_1, A_1) \simeq C^\bullet_{mod}(A, A) \oplus k$. Then, similarly to the Theorem 12.3.1 one proves the following result.

PROPOSITION 12.3.4. *The pair* $(C_\bullet^{mod}(A, A), C^\bullet_{mod}(A, A))$ *is an algebra over the colored operad which is an extension of* $Chains(C_2^{col})$ *by null-ary operations on Hochschild chain and cochain complexes, which correspond to the unit in* A*, and such that for* $h = 0, n = 0$ *and coinciding points in = out, the corresponding operation is the identity.*

12.4. Remark about Gauss-Manin connection. Let $R = k[[t_1, ..., t_n]]$ be the algebra of formal series, and A be an R-flat A_∞-algebra. Then the (modified) negative cyclic complex $CC_\bullet^{-,mod}(A) = (C_\bullet(A, A)[[u]], b + uB)$ is an $R[[u]]$-module. It follows from the existense of Gauss-Manin connection (see [Get]) that the cohomology $HC_\bullet^{-,mod}(A)$ is in fact a module over the ring

$$D_R(A) := k[[t_1, ..., t_n, u]][u\partial/\partial t_1, ..., u\partial/\partial t_n].$$

Inedeed, if ∇ is the Gauss-Manin connection from [Get] then $u\partial/\partial t_i$ acts on the cohomology as $u\nabla_{\partial/\partial t_i}, 1 \leq i \leq n$.

The above considerations can be explained from the point of view of conjecture below. Let $g = C^\bullet(A, A)[1]$ be the DGLA associated with the Hochschild cochain complex, and $M := (C_\bullet^{-,mod}(A)$. We define a DGLA $\hat{g}$ which is the crossproduct $(g \otimes k\langle\xi\rangle) \rtimes k(\partial/\partial\xi)$, where $deg\, \xi = +1$.

CONJECTURE 12.4.1. There is a structure of an L_∞-module on M over $\hat{g}$ which extends the natural structure of a g-module and such that $\partial/\partial\xi$ acts as Connes differential B. Moreover this structure should follow from the P-algebra structure described in Section 12.3.

It looks plausible that the formulas for the Gauss-Manin connection from [Get] can be derived from our generalization of Deilgne's conjecture. We will discuss flat connections on periodic cyclic homology later in the text.

12.5. Flat connections and the colored operad. We start with **Z**-graded case. Let us interpret the **Z**-graded formal scheme $Spf(k[[u]])$ as even formal line equipped with the $\mathbf{G}_m$-action $u \mapsto \lambda^2 u$. The space $HC_\bullet^{-,mod}(A)$ can be interpreted as a space of sections of a $\mathbf{G}_m$-equivariant vector bundle ξ_A over $Spf(k[[u]])$ corresponding to the $k[[u]]$-flat module $\varprojlim_n H^\bullet(C_\bullet^{(n)}(A,A))$. The action of $\mathbf{G}_m$ identifies fibers of this vector bundle over $u \neq 0$. Thus we have a natural flat connection ∇ on the restriction of ξ_A to the complement of the point 0 which has the pole of order one at $u = 0$.

Here we are going to introduce a different construction of the connection ∇ which works also in $\mathbf{Z/2}$-graded case. This connection will have in general a pole of degree two at $u = 0$. In particular we have the following result.

PROPOSITION 12.5.1. *The space of section of the vector bundle ξ_A can be endowed with a structure of a $k[[u]][[u^2\partial/\partial u]]$-module.*

In fact we are going to give an explicit construction of the connection, which is based on the action of the colored dg-operad P discussed in Section 12.3 (more precisely, an extension P^{new} of P, see below). Before presenting an explicit formula, we will make few comments.

1. For any $\mathbf{Z/2}$-graded A_∞-algebra A one can define canonically a 1-parameter family of A_∞-algebras $A_\lambda, \lambda \in \mathbf{G}_m$, such that $A_\lambda = A$ as a $\mathbf{Z/2}$-graded vector space and $m_n^{A_\lambda} = \lambda m_n^A$.

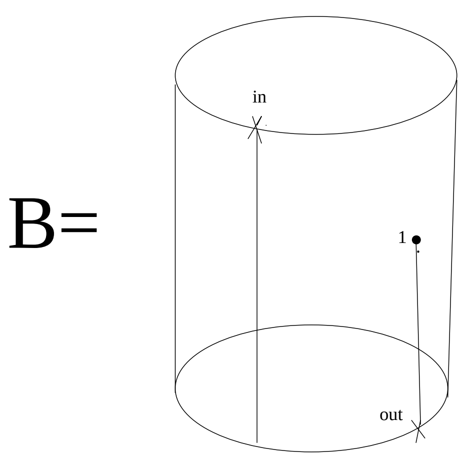
2. For simplicity we will assume that A is strictly unital. Otherwise we will work with the pair $(C_\bullet^{mod}(A,A), C^\bullet_{mod}(A,A))$ of modified Hochschild complexes.

3. We can consider an extension P^{new} of the dg-operad P allowing any non-zero valency for a non-labeled (black) vertex(in the definition of P we required that such a valency was at least three). All the formulas remain the same. But the dg-operad P^{new} is no longer formal. It contains a dg-suboperad generated by trees with all vertices being non-labeled. Action of this suboperad P^{new}_{nonl} is responsible for the flat connection discussed below.

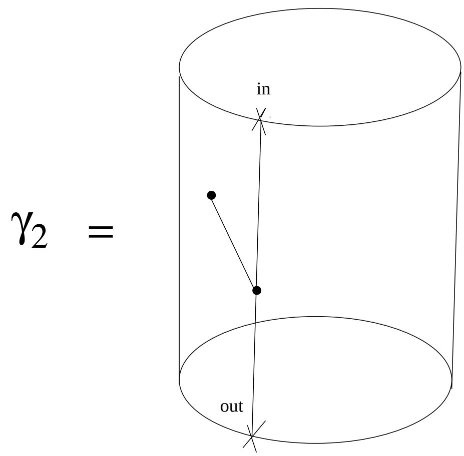
4. In addition to the connection along the variable u one has the Gauss-Manin connection which acts along the fibers of ξ_A (see Section 12.4). Probably one can write down an explicit formula for this connection using the action of the colored operad P^{new}. In what follows are going to describe a connection which presumably coincides with the Gauss-Manin connection.

Let us now consider a dg-algebra $k[B, \gamma_0, \gamma_2]$ which is generated by the following operations of the colored dg-operad P^{new}:

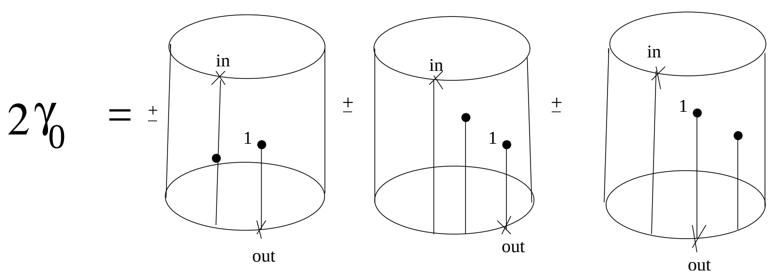
a) Connes differential B of degree -1. It can be depicted such as follows (cf. Section 8.3):



b) Generator γ_2 of degree 2, corresponding to the following figure:



c) Generator γ_0 of degree 0, where $2\gamma_0$ is depicted below:



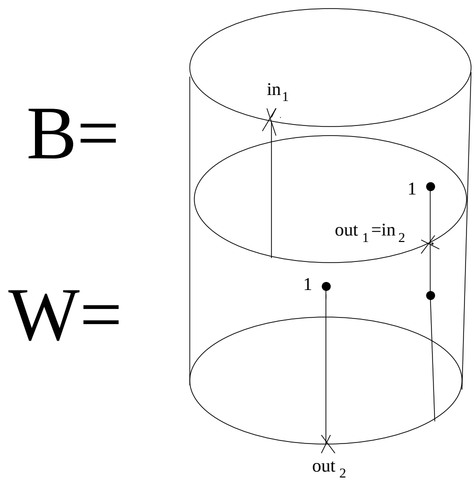
Proposition 12.5.2. *The following identities hold in* P^{new}*:*

$$B^2 = dB = d\gamma_2 = 0, d\gamma_0 = [B, \gamma_2],$$

$$B\gamma_0 + \gamma_0 B := [B, \gamma_0]_+ = -B.$$

Here by d we denote the Hochschild chain differential (previously it was denoted by b).

Proof. Let us prove that $[B, \gamma_0] = -B$, leaving the rest as an exercise to the reader. One has the following identities for the compositions of operations in P^{new}: $B\gamma_0 = 0$, $\gamma_0 B = B$. Let us check, for example, the last identity. Let us denote by W the first summand on the figure defining $2\gamma_0$. Then $\gamma_0 B = \frac{1}{2}WB$. The latter can be depicted in the following way:



It is easily seen equals to $2 \cdot \frac{1}{2} B = B$. ■

COROLLARY 12.5.3. *Hochschild chain complex $C_\bullet(A,A)$ is a dg-module over the dg-algebra $k[B,\gamma_0,\gamma_2]$.*

Let us consider the truncated negative cyclic complex $(C_\bullet(A,A)[[u]]/(u^n), d_u = d + uB)$. We introduce a k-linear map ∇ of $C_\bullet(A,A)[[u]]/(u^n)$ into itself such that $\nabla_{u^2\partial/\partial u} = u^2\partial/\partial u - \gamma_2 + u\gamma_0$. Then we have:

a) $[\nabla_{u^2\partial/\partial u}, d_u] = 0$;

b) $[\nabla_{u^2\partial/\partial u}, u] = u^2$.

Let us denote by V the unital dg-algebra generated by $\nabla_{u^2\partial/\partial u}$ and u, subject to the relations a), b) and the relation $u^n = 0$. From a) and b) one deduces the following result.

PROPOSITION 12.5.4. *The complex $(C_\bullet(A,A)[[u]]/(u^n), d_u = d + uB)$ is a V-module. Moreover, assuming the degeneration conjecture, we see that the operator $\nabla_{u^2\partial/\partial u}$ defines a flat connection on the cohomology bundle*

$$H^\bullet(C_\bullet(A,A)[[u]]/(u^n), d_u)$$

which has the only singularity at $u = 0$ which is a pole of second order.

Taking the inverse limit over n we see that $H^\bullet(C_\bullet(A,A)[[u]], d_u)$ gives rise to a vector bundle over $\mathbf{A}^1_{form}[-2]$ which carries a flat connection with the second order pole at $u = 0$. It is interesting to note the difference between $\mathbf{Z}$-graded and $\mathbf{Z/2}$-graded A_∞-algebras. It follows from the explicit formula for the connection ∇ that the coefficient of the second degree pole is represented by multiplication by a cocyle $(m_n)_{n\geq 1} \in C^\bullet(A,A)$. In cohomology it is trivial in $\mathbf{Z}$-graded case (because of the invariance with respect to the group action $m_n \mapsto \lambda\, m_n$), but nontrivial in $\mathbf{Z/2}$-graded case. Therefore the order of the pole of ∇ is equal to one for $\mathbf{Z}$-graded A_∞-algebras and is equal to two for $\mathbf{Z/2}$-graded A_∞-algebras. We see that in $\mathbf{Z}$-graded case the connection along the variable u comes from the action of the group $\mathbf{G}_m$ on higher products m_n, while in $\mathbf{Z/2}$-graded case it is more complicated.

12.6. PROP of marked Riemann surfaces. In this secttion we will describe a PROP naturally acting on the Hochschild complexes of a finite-dimensional A_∞-algebra with the scalar product of degree N.

Since we have a quasi-isomorphism of complexes

$$C^\bullet(A,A) \simeq (C_\bullet(A,A))^*[-N]$$

it suffices to consider the chain complex only.

In this subsection we will assume that A is either $\mathbf{Z}$-graded (then N is an integer) or $\mathbf{Z/2}$-graded (then $N \in \mathbf{Z/2}$). We will present the results for non-unital A_∞-algebras. In this case we will consider the modified Hochschild chain complex

$$C_\bullet^{mod} = \oplus_{n\geq 0} A \otimes (A[1])^{\otimes n} \bigoplus \oplus_{n\geq 1}(A[1])^{\otimes n},$$

equipped with the Hochschild chain differential (see Section 7.4).

Our construction is summarized in i)-ii) below.

i) Let us consider the topological PROP $\mathcal{M} = (\mathcal{M}(n,m))_{n,m\geq 0}$ consisting of moduli spaces of metrics on compacts oriented surfaces with bondary consisting of $n+m$ circles and some additional marking (see precise definition below).

ii) Let $Chains(\mathcal{M})$ be the corresponding PROP of singular chains. Then there is a structure of a $Chains(\mathcal{M})$-algebra on $C_\bullet^{mod}(A,A)$, which is encoded in a collection of morphisms of complexes

$$Chains(\mathcal{M}(n,m)) \otimes C_\bullet^{mod}(A,A)^{\otimes n} \to (C_\bullet^{mod}(A,A))^{\otimes m}.$$

In addition one has the following:

iii) If A is homologically smooth and satisfies the degeneration property then the structure of $Chains(\mathcal{M})$-algebra extends to a structure of a $Chains(\overline{\mathcal{M}})$-algebra, where $\overline{\mathcal{M}}$ is the topological PROP of stable compactifications of $\mathcal{M}(n,m)$.

Definition 12.6.1. An element of $\mathcal{M}(n,m)$ is an isomorphism class of triples $(\Sigma, h, mark)$ where Σ is a compact oriented surface (not necessarily connected) with metric h, and $mark$ is an orientation preserving isometry between a neighborhood of $\partial\Sigma$ and the disjoint union of $n+m$ flat semiannuli $\sqcup_{1\leq i\leq n}(S^1\times[0,\varepsilon))\sqcup\sqcup_{1\leq i\leq m}(S^1\times[-\varepsilon,0])$, where ε is a sufficiently small positive number. We will call n circle "inputs" and the rest m circles "outputs". We will assume that each connected component of Σ has at least one input, and there are no discs among the connected components. Also we will add $\Sigma = S^1$ to $\mathcal{M}(1,1)$ as the identity morphism. It can be thought of as the limit of cylinders $S^1\times[0,\varepsilon]$ as $\varepsilon\to 0$.

The composition is given by the natural gluing of surfaces.

Let us describe a construction of the action of $Chains(\mathcal{M})$ on the Hochschild chain complex. In fact, instead of $Chains(\mathcal{M})$ we will consider a quasi-isomorphic dg-PROP $R = (R(n,m)_{n,m\geq 0})$ generated by ribbon graphs with additional data. In what follows we will skip some technical details in the definition of the PROP R. They can be recovered in a more or less straightforward way.

It is well-known (and can be proved with the help of Strebel differentials) that $\mathcal{M}(n,m)$ admits a stratification with strata parametrized by graphs described below. More precisely, we consider the following class of graphs.

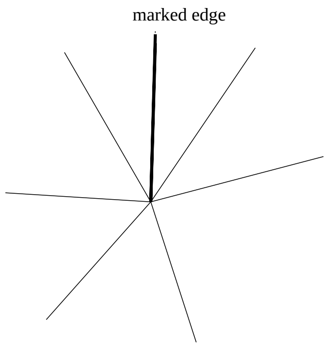
1) Each graph Γ is a (not necessarily connected) ribbon graph (i.e. we are given a cyclic order on the set $Star(v)$ of edges attached to a vertex v of Γ). It is well-known that replacing an edge of a ribbon graph by a thin stripe (thus getting a "fat graph") and gluing stripes in the cyclic order one gets a Riemann surface with the boundary.

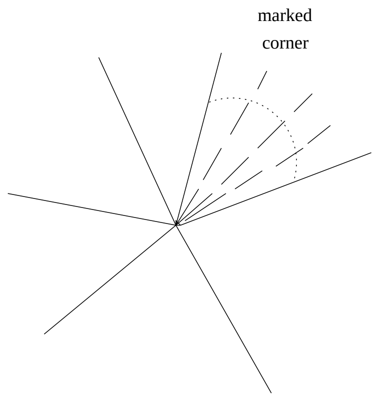
2) The set $V(\Gamma)$ of vertices of Γ is the union of three sets: $V(\Gamma) = V_{in}(\Gamma)\cup V_{middle}(\Gamma)\cup V_{out}(\Gamma)$. Here $V_{in}(\Gamma)$ consists of n numbered vertices $in_1, ..., in_n$ of the

valency 1 (the outcoming edges are called tails), $V_{middle}(\Gamma)$ consists of vertices of the valency greater or equal than 3, and $V_{out}(\Gamma)$ consists of m numbered vertices $out_1, ..., out_m$ of valency greater or equal than 1.

3) We assume that the Riemann surface corresponding to Γ has n connected boundary components each of which has exactly one input vertex.

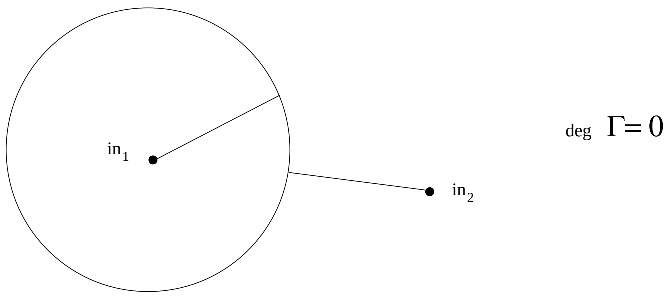
4) For every vertex $out_j \in V_{out}(\Gamma), 1 \leq j \leq m$ we *mark* either an incoming edge or a pair of adjacent (we call such a pair of edges a *corner*).

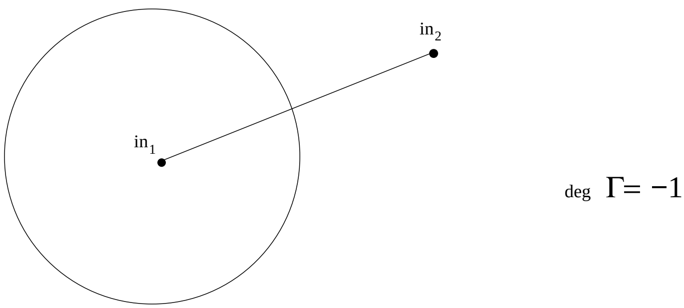




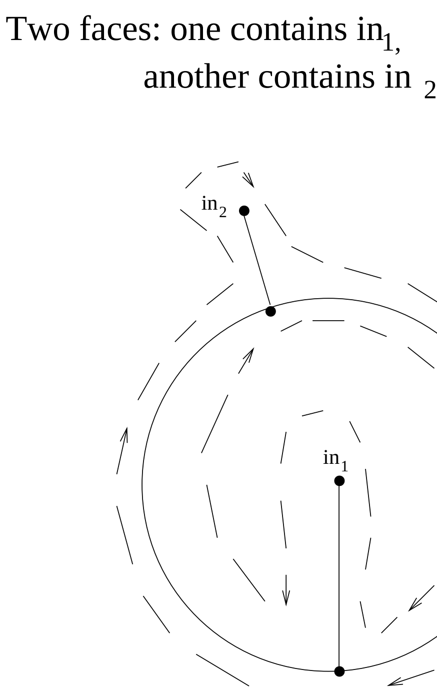
More pedantically, let $E(\Gamma)$ denotes the set of edges of Γ and $E^{or}(\Gamma)$ denotes the set of pairs (e, or) where $e \in E(\Gamma)$ and or is one of two possible orientations of e. There is an obvious map $E^{or}(\Gamma) \to V(\Gamma) \times V(\Gamma)$ which assigns to an oriented edge the pair of its endpoint vertices: source and target. The free involution σ acting on $E^{or}(\Gamma)$ (change of orientation) corresponds to the permutation map on $V(\Gamma) \times V(\Gamma)$. Cyclic order on each $Star(v)$ means that there is a bijection $\rho : E^{or}(\Gamma) \to E^{or}(\Gamma)$ such that orbits of iterations $\rho^n, n \geq 1$ are elements of $Star(v)$ for some $v \in V(\Gamma)$. In particular, the corner is given either by a pair of coinciding edges (e, e) such that $\rho(e) = e$ or by a pair edges $e, e' \in Star(v)$ such that $\rho(e) = e'$. Let us define a *face* as an orbit of $\rho \circ \sigma$. Then faces are oriented closed paths. It follows from the condition 2) that each face contains exactly one edge outcoming from some in_i.

We depict below two graphs in the case $g = 0, n = 2, m = 0$.



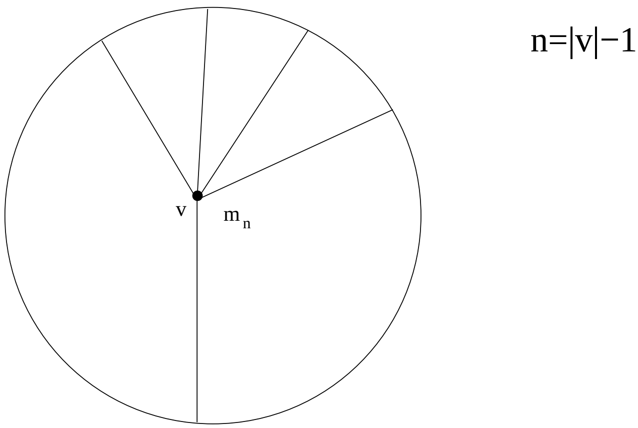


Here is a picture illustrating the notion of face



REMARK 12.6.2. The above data (i.e. a ribbon graph with numerations of *in* and *out* vertices) have no automorphisms. Thus we can identify Γ with its isomorphism class.

The functional $(m_n(a_1, ..., a_n), a_{n+1})$ is depicted such as follows.



We define the *degree* of Γ by the formula

$$deg\,\Gamma = \sum_{v\in V_{middle}(\Gamma)} (3-|v|) + \sum_{v\in V_{out}(\Gamma)} (1-|v|) + \sum_{v\in V_{out}(\Gamma)} \epsilon_v - N\chi(\Gamma),$$

where $\epsilon_v = -1$, if v contains a marked corner and $\epsilon_v = 0$ otherwise. Here $\chi(\Gamma) = |V(\Gamma)| - |E(\Gamma)|$ denotes the Euler characteristic of Γ.

DEFINITION 12.6.3. We define $R(n, m)$ as a graded vector space which is a direct sum $\oplus_\Gamma \psi_\Gamma$ of 1-dimensional graded vector spaces generated by graphs Γ as above, each summand has degree $deg\,\Gamma$.

One can see that ψ_Γ is naturally identified with the tensor product of 1-dimensional vector spaces (determinants) corresponding to vertices of Γ.

Now, having a graph Γ which satisfies conditions 1-3) above, and Hochschild chains $\gamma_1, ..., \gamma_n \in C_\bullet^{mod}(A, A)$ we would like to define an element of $C_\bullet^{mod}(A, A)^{\otimes m}$. Roughly speaking we are going to assign the above n elements of the Hochschild complex to n faces corresponding to vertices $in_i, 1 \leq i \leq n$, then assign tensors corresponding to higher products m_l to internal vertices $v \in V_{middle}(\Gamma)$, then using the convolution operation on tensors given by the scalar product on A to read off the resulting tensor from $out_j, 1 \leq j \leq m$. More precise algorithm is described below.

a) We decompose the modified Hochschild complex such as follows:

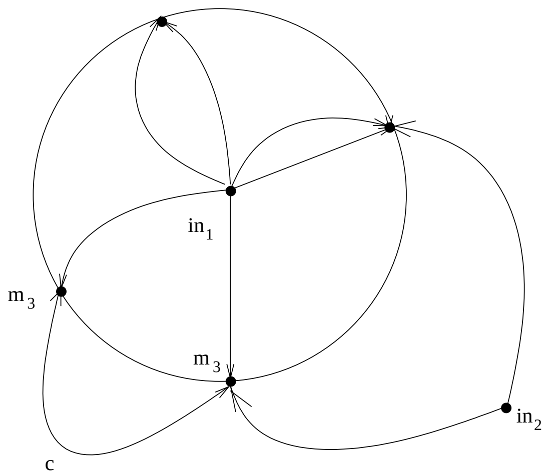
$$C_\bullet^{mod}(A, A) = \oplus_{l\geq 0, \varepsilon\in\{0,1\}} C_{l,\varepsilon}^{mod}(A, A),$$

where $C_{l,\varepsilon=0}^{mod}(A, A) = A \otimes (A[1])^{\otimes l}$ and $C_{l,\varepsilon=1}^{mod}(A, A) = k \otimes (A[1])^{\otimes l}$ according to the definition of modified Hochschild chain complex. For any choice of $l_i \geq 0, \varepsilon_i \in \{0, 1\}, 1 \leq i \leq n$ we are going to construct a linear map of degree zero

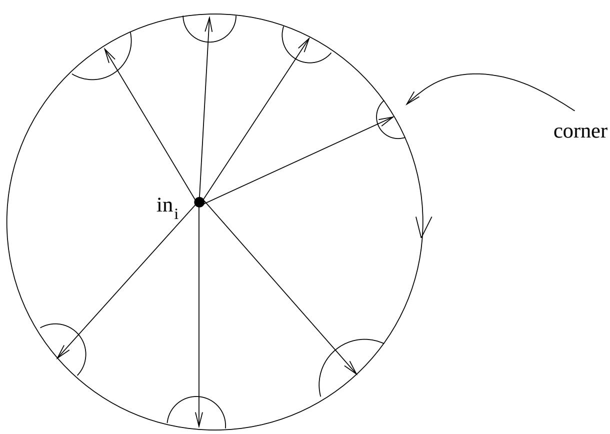
$$f_\Gamma : \psi_\Gamma \otimes C_{l_1,\varepsilon_1}^{mod}(A, A) \otimes ... \otimes C_{l_n,\varepsilon_1}^{mod}(A, A) \to (C_\bullet^{mod}(A, A))^{\otimes m}.$$

The result will be a sum $f_\Gamma = \sum_{\Gamma'} f_{\Gamma'}$ of certain maps. The description of the collection of graphs Γ' is given below.

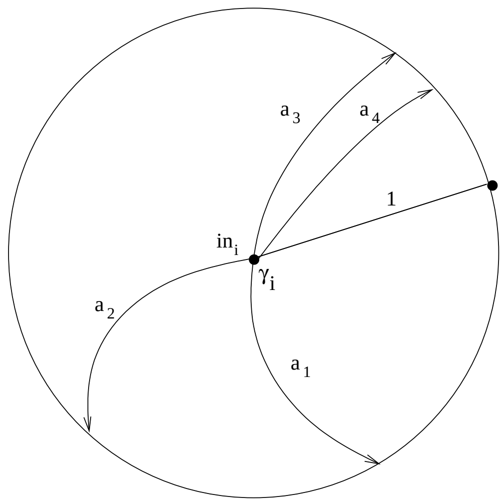
b) Each new graph Γ' is obtained from Γ by adding new edges. More precisely one has $V(\Gamma') = V(\Gamma)$ and for each vertex $in_i \in V_{in}(\Gamma)$ we add l_i new outcoming edges. Then the valency of in_i becomes $l_i + 1$.



More pedantically, for every $i, 1 \leq i \leq n$ we have constructed a map from the set $\{1, ..., l_i\}$ to a cyclically ordered set which is an orbit of $\rho \circ \sigma$ with removed the tail edge oucoming from in_i. Cyclic order on the edges of Γ' is induced by the cyclic order at every vertex and the cyclic order on the path forming the face corresponding to in_i.



c) We assign $\gamma_i \in C_{l_i,\varepsilon_i}$ to in_i. We depict γ_i as a "wheel" representing the Hochschild cocycle. It is formed by the endpoints of the $l_i + 1$ edges outcoming from $in_i \in V(\Gamma')$ and taken in the cyclic order of the corresponding face. If $\varepsilon_i = 1$ then (up to a scalar) $\gamma_i = 1 \otimes a_1 \otimes ... \otimes a_{l_i}$, and we require that the tensor factor 1 corresponds to zero in the cyclic order.



d) We remove from considerations graphs Γ which do not obey the following property after the step c):

the edge corresponding to the unit $1 \in k$ *(see step c)) is of the type* (in_i, v) *where either* $v \in V_{middle}(\Gamma')$ *and* $|v| = 3$ *or* $v = out_j$ *for some* $1 \le j \le m$ *and the edge* (in_i, out_j) *was the marked edge for* out_j.

Let us call *unit edge* the one which satisfies one of the above properties. We define a new graph Γ'' which is obtained from Γ by removing unit edges.

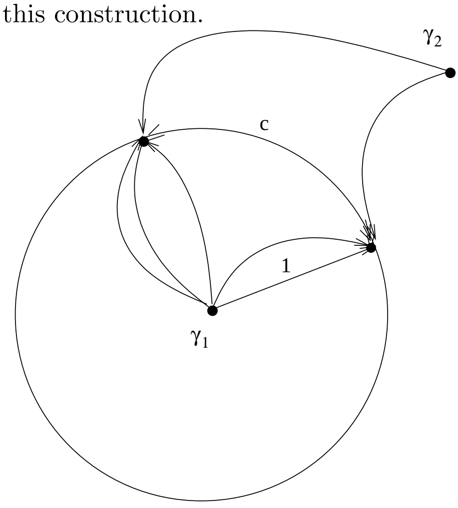
e) Each vertex now has the valency $|v| \ge 2$. We attach to every such vertex either:

the tensor $c \in A \otimes A$ (inverse to the scalar product), if $|v| = 2$,

or

the tensor $(m_{|v|-1}(a_1, ..., a_{|v|-1}), a_{|v|})$ if $|v| \ge 3$. The latter can be identified with the element of $A^{\otimes |v|}$ (here we use the non-degenerate scalar product on A).

Let us illustrate this construction.



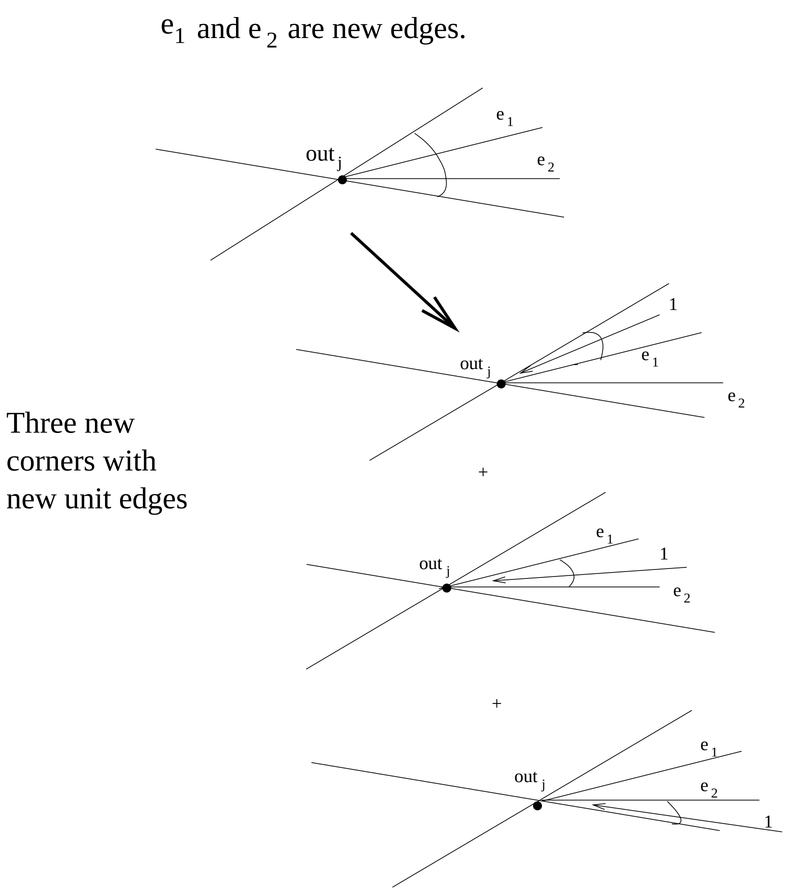
f) Let us contract indices of tensors corresponding to $V_{in}(\Gamma'') \cup V_{middle}(\Gamma'')$ (see c), e)) along the edges of Γ'' using the scalar product on A. The result will be an element a_{out} of the tensor product $\otimes_{1\le j\le m} A^{Star_{\Gamma''}(out_j)}$.

g) Last thing we need to do is to interpret the element a_{out} as an element of $C^{mod}_{\bullet}(A,A)$. There are three cases.

Case 1. When we constructed Γ'' there was a unit edge incoming to some out_j. Then we reconstruct back the removed edge, attach $1 \in k$ to it, and interpret the resulting tensor as an element of $C^{mod}_{|out_j|,\varepsilon_j=1}(A,A)$.

Case 2. There was no removed unit edge incoming to out_j and we had a marked edge (not a marked corner) at the vertex out_j. Then we have an honest element of $C^{mod}_{|out_j|,\varepsilon_j=0}(A,A)$

Case 3. Same as in Case 2, but there was a marked corner at $out_j \in V_{out}(\Gamma)$. We have added and removed new edges when constructed Γ''. Therefore the marked corner gives rise to a new set of marked corners at out_j considered as a vertex of Γ''. Inside every such a corner we insert a new edge, attach the element $1 \in k$ to it and take the sum over all the corners. In this way we obtain an element of $C^{mod}_{|out_j|,\varepsilon_j=1}(A,A)$. This procedure is depicted below.



This concludes the construction of f_Γ. Notice that R is a dg-PROP with the differential given by the insertion of a new edge between two vertices from $V_{middle}(\Gamma)$.

Proof of the following Proposition will be given elsewhere.

PROPOSITION 12.6.4. *The above construction gives rise to a structure of a R-algebra on $C_\bullet^{mod}(A, A)$.*

REMARK 12.6.5. The above construction did not use homological smoothness of A.

Finally we would like to say few words about an extension of the R-action to the $Chains(\overline{\mathcal{M}})$-action.

If we assume the degeneration property for A, then the action of the PROP R can be extended to the action of the PROP $Chains(\overline{\mathcal{M}})$ of singular chains of the topological PROP of stable degenerations of $M_{g,n,m}^{marked}$. In order to see this, one introduces the PROP D freely generated by $R(2, 0)$ and $R(1, 1)$, i.e. by singular chains on the moduli space of cylinders with two inputs and zero outputs (they correspond to the scalar product on $C_\bullet(A, A)$) and by cylinders with one input and one output (they correspond to morphisms $C_\bullet(A, A) \to C_\bullet(A, A)$). In fact the (non-symmetric) bilinear form $h : H_\bullet(A, A) \otimes H_\bullet(A, A) \to k$ does exist for any compact A_∞-algebra A. It is described by the graph of degree zero on the figure in Section 12.6. This is a generalization of the bilinear form $(a, b) \in A/[A, A] \otimes A/[A, A] \mapsto Tr(axb) \in k$. It seems plausible that homological smoothness implies that h is non-degenerate. This allows us to extend the action of the dg sub-PROP $D \subset R$ to the action of the dg PROP $D' \subset R$ which contains also $R(0, 2)$ (i.e. the inverse to the above bilinear form). If we assume the degeneration property, then we can "shrink" the action of the homologically non-trivial circle of the cylinders (since the rotation around this circle corresponds to the differential B). Thus D' is quasi-isomorphic to the dg-PROP of chains on the (one-dimensional) retracts of the above cylinders (retraction contracts the circle). Let us denote the dg-PROP generated by singular chains on the retractions by D''. Thus, assuming the degeneration property, we see that the free product dg-PROP $R' = R *_D D''$ acts on $C_\bullet^{mod}(A, A)$. One can show that R' is quasi-isomorphic to the dg-PROP of chains on the topological PROP $\overline{M}_{g,n,m}^{marked}$ of stable compactifications of the surfaces from $M_{g,n,m}^{marked}$.

REMARK 12.6.6. a) The above construction is generalization of the construction from [Ko92], which assigns cohomology classes of $M_{g,n}$ to a finite-dimensional A_∞-algebra with scalar product (trivalent graphs were used in [Ko92]).

b) Different approach to the action of the PROP R was suggested in [Cos04]. The above Proposition gives rise to a structure of Topological Field Theory associated with a non-unital A_∞-algebra with scalar product. If the degeneration property holds for A then one can define a Cohomological Field Theory in the sense of [KoM94]

c) Homological smoothness of A is closely related to the existence of a non-commutative analog of the Chern class of the diagonal $\Delta \subset X \times X$ of a projective scheme X. This Chern class gives rise to the inverse to the scalar product on A. This topic will be discussed in the subsequent paper devoted to A_∞-categories.

13. Appendix

13.1. Non-commutative schemes and ind-schemes. Let $\mathcal{C}$ be an Abelian k-linear tensor category. To simplify formulas we will assume that it is strict (see [McL71]). We will also assume that $\mathcal{C}$ admits infinite sums. To simplify the exposition we will assume below (and in the main body of the paper) that $\mathcal{C} = Vect_k^{\mathbf{Z}}$.

DEFINITION 13.1.1. The category of non-commutative affine k-schemes in $\mathcal{C}$ (notation $NAff_{\mathcal{C}}$) is the one opposite to the category of associative unital k-algebras in $\mathcal{C}$.

The non-commutative scheme corresponding to the algebra A is denoted by $Spec(A)$. Conversely, if X is a non-commutative affine scheme then the corresponding algebra (algebra of regular functions on X) is denoted by $\mathcal{O}(X)$. By analogy with commutative case we call a morphism $f : X \to Y$ a *closed embedding* if the corresponding homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an epimorphism.

Let us recall some terminology of ind-objects (see for ex. [Gr59], [AM64], [KSch01]). For a covariant functor $\phi : I \to \mathcal{A}$ from a small filtering category I (called filtrant in [KSch01]) there is a notion of an inductive limit "$\varinjlim$"$\phi \in \widehat{\mathcal{A}}$ and a projective limit "$\varprojlim$"$\phi \in \widehat{\mathcal{A}}$. By definition "$\varinjlim$"$\phi(X) = \varinjlim Hom_{\mathcal{A}}(X, \phi(i))$ and "$\varprojlim$"$\phi(X) = \varinjlim Hom_{\mathcal{A}}(\phi(i), X)$. All inductive limits form a full subcategory $Ind(\mathcal{A}) \subset \widehat{\mathcal{A}}$ of ind-objects in $\mathcal{A}$. Similarly all projective limits form a full subcategory $Pro(\mathcal{A}) \subset \widehat{\mathcal{A}}$ of pro-objects in $\mathcal{A}$.

DEFINITION 13.1.2. Let I be a small filtering category, and $F : I \to NAff_{\mathcal{C}}$ a covariant functor. We say that "$\varinjlim$"F is a non-commutative ind-affine scheme if for a morphism $i \to j$ in I the corresponding morphism $F(i) \to F(j)$ is a closed embedding.

In other words a non-commutative ind-affine scheme X is an object of $Ind(NAff_{\mathcal{C}})$, corresponding to the projective limit $\varprojlim A_\alpha, \alpha \in I$, where each A_α is a unital associative algebra in $\mathcal{C}$, and for a morphism $\alpha \to \beta$ in I the corresponding homomorphism $A_\beta \to A_\alpha$ is a surjective homomorphism of unital algebras (i.e. one has an exact sequence $0 \to J \to A_\beta \to A_\alpha \to 0$).

REMARK 13.1.3. Not all categorical epimorphisms of algebras are surjective homomorphisms (although the converse is true). Nevertheless one can define closed embeddings of affine schemes for an arbitrary Abelian k-linear category, observing that a surjective homomorphism of algebras $f : A \to B$ is characterized categorically by the condition that B is the cokernel of the pair of the natural projections $f_{1,2} : A \times_B A \to A$ defined by f.

Morphisms between non-commutative ind-affine schemes are defined as morphisms between the corresponding projective systems of unital algebras. Thus we have

$$Hom_{NAff_{\mathcal{C}}}(\varinjlim{}_I X_i, \varinjlim{}_J Y_j) = \varprojlim{}_I \varinjlim{}_J Hom_{NAff_{\mathcal{C}}}(X_i, Y_j).$$

Let us recall that an algebra $M \in Ob(\mathcal{C})$ is called nilpotent if the natural morphism $M^{\otimes n} \to M$ is zero for all sufficiently large n.

DEFINITION 13.1.4. A non-commutative ind-affine scheme $\hat{X}$ is called formal if it can be represented as $\hat{X} = \varinjlim Spec(A_i)$, where $(A_i)_{i \in I}$ is a projective system of associative unital algebras in $\mathcal{C}$ such that the homomorphisms $A_i \to A_j$ are surjective and have nilpotent kernels for all morphisms $j \to i$ in I.

Let us consider few examples in the case when $\mathcal{C} = Vect_k$.

EXAMPLE 13.1.5. In order to define the non-commutative formal affine line $\widehat{\mathbf{A}}^1_{NC}$ it suffices to define $Hom(Spec(A), \widehat{\mathbf{A}}^1_{NC})$ for any associative unital algebra A. We define $Hom_{NAff_k}(Spec(A), \widehat{\mathbf{A}}^1_{NC}) = \varinjlim Hom_{Alg_k}(k[[t]]/(t^n), A)$. Then the set of A-points of the non-commutative formal affine line consists of all nilpotent elements of A.

EXAMPLE 13.1.6. For an arbitrary set I the non-commutative formal affine space $\widehat{\mathbf{A}}^I_{NC}$ corresponds, by definition, to the topological free algebra $k\langle\langle t_i \rangle\rangle_{i \in I}$. If A is a unital k-algebra then any homomorphism $k\langle\langle t_i \rangle\rangle_{i \in I} \to A$ maps almost all t_i to zero, and the remaining generators are mapped into nilpotent elements of A. In particular, if $I = \mathbf{N} = \{1, 2, ...\}$ then $\widehat{\mathbf{A}}^{\mathbf{N}}_{NC} = \varinjlim Spec(k\langle\langle t_1, ..., t_n \rangle\rangle/(t_1, ..., t_n)^m)$, where $(t_1, ..., t_n)$ denotes the two-sided ideal generated by $t_i, 1 \leq i \leq n$, and the limit is taken over all $n, m \to \infty$.

By definition, a *closed subscheme* Y of a scheme X is defined by a 2-sided ideal $J \subset \mathcal{O}(X)$. Then $\mathcal{O}(Y) = \mathcal{O}(X)/J$. If $Y \subset X$ is defined by a 2-sided ideal $J \subset \mathcal{O}(X)$, then the completion of X along Y is a formal scheme corresponding to the projective limit of algebras $\varprojlim_n \mathcal{O}(X)/J^n$. This formal scheme will be denoted by $\hat{X}_Y$ or by $Spf(\mathcal{O}(X)/J)$.

Non-commutative affine schemes over a given field k form symmetric monoidal category. The tensor structure is given by the *ordinary tensor product* of unital algebras. The corresponding tensor product of non-commutative affine schemes will be denoted by $X \otimes Y$. It is not a categorical product, differently from the case of commutative affine schemes (where the tensor product of algebras corresponds to the Cartesian product $X \times Y$). For non-commutative affine schemes the analog of the Cartesian product is the *free product* of algebras.

Let A, B be free algebras. Then $Spec(A)$ and $Spec(B)$ are non-commutative manifolds. Since the tensor product $A \otimes B$ in general is not a smooth algebra, the non-commutative affine scheme $Spec(A \otimes B)$ is not a manifold.

Let X be a non-commutative ind-affine scheme in $\mathcal{C}$. A closed k-point $x \in X$ is by definition a homomorphism of $\mathcal{O}(X)$ to the tensor algebra generated by the unit object $\mathbf{1}$. Let m_x be the kernel of this homomorphism. We define the *tangent space* T_xX in the usual way as $(m_x/m_x^2)^* \in Ob(\mathcal{C})$. Here m_x^2 is the image of the multiplication map $m_x^{\otimes 2} \to m_x$.

A non-commutative ind-affine scheme with a marked closed k-point will be called *pointed*. There is a natural generalization of this notion to the case of many points. Let $Y \subset X$ be a closed subscheme of disjoint closed k-points (it corresponds to the algebra homomorphism $\mathcal{O}(X) \to \mathbf{1} \oplus \mathbf{1} \oplus ...$). Then $\hat{X}_Y$ is a formal manifold. A pair $(\hat{X}_Y, Y)$ (often abbreviated by $\hat{X}_Y$) will be called (non-commutative) *formal manifold with marked points*. If Y consists of one such point then $(\hat{X}_Y, Y)$ will be called (non-commutative) *formal pointed manifold*.

13.2. Proof of Theorem 2.1.2. In the category $Alg_{\mathcal{C}^f}$ every pair of morphisms has a kernel. Since the functor F is left exact and the category $Alg_{\mathcal{C}^f}$ is Artinian, it follows from [Gr59], Sect. 3.1 that F is strictly pro-representable. This means that there exists a projective system of finite-dimensional algebras $(A_i)_{i \in I}$ such that, for any morphism $i \to j$ the corresponding morphsim $A_j \to A_i$ is a categorical epimorphism, and for any $A \in Ob(Alg_{\mathcal{C}^f})$ one has

$$F(A) = \varinjlim_I Hom_{Alg_{\mathcal{C}^f}}(A_i, A).$$

Equivalently,

$$F(A) = \varinjlim_I Hom_{Coalg_{\mathcal{C}^f}}(A_i^*, A^*),$$

where $(A_i^*)_{i\in I}$ is an inductive system of finite-dimensional coalgebras and for any morphism $i \to j$ in I we have a categorical monomorphism $g_{ji} : A_i^* \to A_j^*$.

All what we need is to replace the projective system of algebras $(A_i)_{i\in I}$ by another projective system of algebras $(\overline{A}_i)_{i\in I}$ such that

a) functors "$\varprojlim$"h_{A_i} and "$\varprojlim$"$h_{\overline{A}_i}$ are isomorphic (here h_X is the functor defined by the formula $h_X(Y) = Hom(X, Y)$);

b) for any morphism $i \to j$ the corresponding homomorphism of algebras $\overline{f}_{ij} : \overline{A}_j \to \overline{A}_i$ is surjective.

Let us define $\overline{A}_i = \bigcap_{i\to j} Im(f_{ij})$, where $Im(f_{ij})$ is the image of the homomorphism $f_{ij} : A_j \to A_i$ corresponding to the morphism $i \to j$ in I. In order to prove a) it suffices to show that for any unital algebra B in $\mathcal{C}^f$ the natural map of sets

$$\varinjlim_I Hom_{\mathcal{C}^f}(A_i, B) \to \varinjlim_I Hom_{\mathcal{C}^f}(\overline{A}_i, B)$$

(the restriction map) is well-defined and bijective.

The set $\varinjlim_I Hom_{\mathcal{C}^f}(A_i, B)$ is isomorphic to $(\bigsqcup_I Hom_{\mathcal{C}^f}(A_i, B))/equiv$, where two maps $f_i : A_i \to B$ and $f_j : A_j \to B$ such that $i \to j$ are equivalent if $f_i f_{ij} = f_j$. Since $\mathcal{C}^f$ is an Artinian category, we conclude that there exists A_m such that $f_{im}(A_m) = \overline{A}_i$, $f_{jm}(A_m) = \overline{A}_j$. From this observation one easily deduces that $f_{ij}(\overline{A}_j) = \overline{A}_i$. It follows that the morphism of functors in a) is well-defined, and b) holds. The proof that morphisms of functors biejectively correspond to homomorphisms of coalgebras is similar. This completes the proof of the theorem. ■

13.3. Proof of Proposition 2.1.3. The result follows from the fact that any $x \in B$ belongs to a finite-dimensional subcoalgebra $B_x \subset B$, and if B was counital then B_x would be also counital. Let us describe how to construct B_x. Let Δ be the coproduct in B. Then one can write

$$\Delta(x) = \sum_i a_i \otimes b_i,$$

where a_i (resp. b_i) are linearly independent elements of B.

It follows from the coassociativity of Δ that

$$\sum_i \Delta(a_i) \otimes b_i = \sum_i a_i \otimes \Delta(b_i).$$

Therefore one can find constants $c_{ij} \in k$ such that

$$\Delta(a_i) = \sum_j a_j \otimes c_{ij},$$

and

$$\Delta(b_i) = \sum_j c_{ji} \otimes b_j.$$

Applying $\Delta \otimes id$ to the last equality and using the coassociativity condition again we get

$$\Delta(c_{ji}) = \sum_n c_{jn} \otimes c_{ni}.$$

Let B_x be the vector space spanned by x and all elements a_i, b_i, c_{ij}. Then B_x is the desired subcoalgebra. ■

13.4. Formal completion along a subscheme. Here we present a construction which generalizes the definition of a formal neighborhood of a k-point of a non-commutative smooth thin scheme.

Let $X = Spc(B_X)$ be such a scheme and $f : X \to Y = Spc(B_Y)$ be a closed embedding, i.e. the corresponding homomorphism of coalgebras $B_X \to B_Y$ is injective. We start with the category $\mathcal{N}_X$ of nilpotent extensions of X, i.e. homomorphisms $\phi : X \to U$, where $U = Spc(D)$ is a non-commutative thin scheme, such that the quotient $D/f(B_X)$ (which is always a non-counital coalgebra) is locally conilpotent. We recall that the local conilpotency means that for any $a \in D/f(B_X)$ there exists $n \geq 2$ such that $\Delta^{(n)}(a) = 0$, where $\Delta^{(n)}$ is the n-th iterated coproduct Δ. If (X, ϕ_1, U_1) and (X, ϕ_2, U_2) are two nilpotent extensions of X then a morphism between them is a morphism of non-commutative thin schemes $t : U_1 \to U_2$, such that $t\phi_1 = \phi_2$ (in particular, $\mathcal{N}_X$ is a subcategory of the naturally defined category of non-commutative relative thin schemes).

Let us consider the functor $G_f : \mathcal{N}_X^{op} \to Sets$ such that $G(X, \phi, U)$ is the set of all morphisms $\psi : U \to Y$ such that $\psi\phi = f$.

PROPOSITION 13.4.1. *Functor G_f is represented by a triple $(X, \pi, \hat{Y}_X)$ where the non-commutative thin scheme denoted by $\widehat{Y}_X$ is called the formal neighborhood of $f(X)$ in Y (or the completion of Y along $f(X)$).*

Proof. Let $B_f \subset B_X$ be the counital subcoalgebra which is the pre-image of the (non-counital) subcoalgebra in $B_Y/f(B_X)$ consisting of locally conilpotent elements. Notice that $f(B_X) \subset B_f$. It is easy to see that taking $\widehat{Y}_X := Spc(B_f)$ we obtain the triple which represents the functor G_f. ■

Notice that $\widehat{Y}_X \to Y$ is a closed embedding of non-commutative thin schemes.

PROPOSITION 13.4.2. *If Y is smooth then $\widehat{Y}_X$ is smooth and $\widehat{Y}_X \simeq \widehat{Y}_{\widehat{Y}_X}$.*

Proof. Follows immediately from the explicit description of the coalgebra B_f given in the proof of the previous Proposition. ■

Bibliography

[AGV] V. Arnold, S. Gusein-Zade, A. Varchenko, Singularities of smooth maps.

[AM64] M. Artin, B. Mazur, Etale Homotopy, Lect. Notes Math., 100 (1969).

[BD03] A. Beilinson, V. Drinfeld, Chiral algebras.

[BD06] A. Beilinson, V. Drinfeld, Quantization of Hitchin's integrable system and Hecke eigensheaves (in preparation).

[BK91] A. Bondal, M. Kapranov, Enhanced triangulated categories, Math. USSR Sbornik, 70:1, 93-107 (1991).

[Bou] N. Bourbaki, Set Theory.

[BV73] J. Boardman, R. Vogt, Homotopy invariant algebraic structures on topological spaces, Lect. Notes Math. 347 (1973).

[BvB02] A. Bondal, M. Van den Bergh Generators and representability of functors in commutative and noncommutative geometry, math.AG/0204218.

[Co94] A. Connes, Non-commutative geometry. Academic Press, 1994.

[Cos04] K. Costello, Topological conformal field theories, Calabi-Yau categories and Hochschild homology, preprint (2004).

[CQ95-1] J. Cuntz, D. Quillen, Cyclic homology and non-singularity, J. of AMS, 8:2(1995), 373-442.

[CQ95-2] J. Cuntz, D. Quillen, Algebra extensions and non-singularity, J. of AMS, 8:2(1995), 251-289.

[De90] P.Deligne, letter to Drinfeld (1990)

[CST] J. Cuntz, G. Skandalis, B. Tsygan, Cyclic homology in noncommutative geometry, Encyclopaedia of Mathematical Sciences, v. 121, Springer Verlag, p. 74-113.

[DI87] P. Deligne, L. Illusie, Relevements modulo p^2 et decomposition du complex de de Rham, Invent. Math. 89(1987), 247-270.

[DGMS75] P. Deligne, Ph. Griffiths, J. Morgan, and D. Sullivan Real homotopy theory of Kähler manifolds, Invent. Math 29 (1975), 245 - 274.

[DM82] P. Deligne, J.S. Milne, Tannakian categories, Lect. Notes Math., 900 (1982), 101-228.

[Dr02] V. Drinfeld, DG quotients of DG categories, math.KT/0210114.

[EK97] P. Etingof, D. Kazhdan, Quantization of Lie bialgebras I and II, Selecta Mathematica, New series, 1997.

[FM94] W. Fulton, R. Macpherson, A compactification of configuration spaces, Annals Math., pages 183-225,139(1994).

[FOOO] K. Fukaya, Y.G. Oh, H. Ohta, K. Ono, Lagrangian intersection Floer theory-anomaly and obstruction. Preprint, 2000.

[Get] E. Getzler, Cartan homotopy formulas and Gauss-Manin connection in cyclic homology, Israel Math. Conf. Proc., 7, 65-78.

[GetJ94] E. Getzler, J. Jones, Operads,homotopy algebra, and iterated integrals for double loop spaces. Preprint 1994.

[GetKa95] E. Getzler, M. Kapranov, Cyclic operads and cyclic homology. In: Geometry, Topology , and Physics for Raul Bott. International Press, 1995, p. 167-201.

[Gi03] V. Ginzburg, Lectures on Noncommutative Geometry,math.AG/0506603.

[Gi05] V. Ginzburg, Double derivations and cyclic homology, math.KT/0505236.

[GiKa94] V. Ginzburg, M. Kapranov, Koszul duality for operads, Duke Math. J. 76 (1994), 203-272.

[Gi2000] V. Ginzburg, Non-commutative symplectic geometry, quiver varieties and operads, math.QA/0005165.

[Gr59] A. Grothendieck, Technique de descente et theoremes d'existence en geometrie algebrique II. Le theoreme d'existence en theorie formelle des modules, Sem. Bourbaki, 195 (1959/60).

[GS88] M. Gerstenhaber and S.Schack, Algebraic cohomology and deformation theory. Kluwer Acad. Publ., Dordrecht, Boston, 1988.

[HKV03] P. Hu, I. Kriz, A. Voronov, On Kontsevich's Hochschild cohomology conjecture, math.AT/0309369.

[KSch01] M. Kashiwara, P. Schapira, Ind-sheaves, Asterisque 271 (2001).

[KaKoP], L.Katzarkov, M. Kontsevich, T. Pantev, Calculating Gromov-Witten invariants from Fukaya category, in preparation.

[Kal05] D. Kaledin, Non-commutative Cartier operator and Hodge-to-de Rham degeneration, math.AG/0511665.

[Kau06] R. Kaufmann,Moduli space actions on the Hochschild co-chains of a Frobenius algebra I: Cell Operads, math.AT/0606064.

[KorSo98] L. Korogodsky, Y. Soibelman, Algebras of functions on quantum groups.I, AMS, 1998.

[Ke01] B. Keller, Introduction to A-infinity algebras and modules, Homology, Homotopy and Applications 3 (2001), 1-35.

[Ke06] B. Keller, On differential graded categories, math.KT/0601185.

[Ko92] M. Kontsevich, Feynman diagrams and low-dimensional topology. Proc. Europ. Congr. Math. vol. 1 (1992).

[Ko94] M. Kontsevich, Homological algebra of mirror symmetry, Proceedings of ICM-94, vol. 1.

[Ko97] M. Kontsevich, Deformation quantization of Poisson manifolds,I. math.QA/9709040

[Ko97-1] M. Kontsevich, Rozansky-Witten invariants via formal geometry. math.QA/9704

[Ko99] M.Kontsevich, Operads and motives in deformation quantization. Letters in Math.Phys., 48:1 (1999), math.QA/9904055

[Ko06] M. Kontsevich, Notes on motives in finite characteristic.

[KoM94] M. Kontsevich, Yu.Manin, Gromov-Witten classes, quantum cohomology and enumerativ geometry, Comm. Math. Phys., 164:3(1994),525-562.

[KoSo2000] M. Kontsevich, Y. Soibelman, Deformations of algebras over operads and Deligne conjecture, math.QA/0001151, published in Lett. Math. Phys. 21:1 (2000), 255-307.

[Le Br] L. Le Bruyn, Non-commutative geometry an n (book in preparation).

[Lyu02] V. Lyubashenko, Category of A_∞-categories, math.CT/0210047.

[LyuOv02] V. Lyubashenko, S. Ovsienko, A construction of A_∞-category, math.CT/021103

[M02] M. Markl, A resolution (minimal model) of the PROP of bialgebras, math.AT/0209007.

[MV03] M. Markl, A. Voronov, PROPped up graph cohomology, math.QA/0307081.

[Ma77] P. May, Infinite loop space theory, Bull. AMS, 83:4(1977),456-494.

[McL71] S. Maclane, Categories for the working mathematician, Springer-Verlag, Berlin, New York, 1971.

[Or05] D. Orlov,Triangulated categories of singularities and equivalences between Landau-Ginzburg models, math.AG/0503630.

[R03] R. Rouquier, Dimensions of triangulated categories, math.CT/0310134.

[Se03] P. Seidel, Homological mirror symmetry for the quartic surface, math.SG/0310414.

[So03] Y. Soibelman, Non-commutative geometry and deformations of A_∞-algebras and A_∞-categories, Arbeitstagung 2003, Preprint Max-PLanck Institut für Mathematik, MPIM2003-60h, 2003.

[So04] Y. Soibelman, Mirror symmetry and non-commutative geometry of A_∞-categories, Journal of Mathematical Physics, vol. 45:10, 2004, 3742-3757

[SU2000] S. Sanablidze, R. Umble, A diagonal of the associahedra, math. AT/0011065.

[T98] D. Tamarkin, Formality of chain operad of small squares, math.QA/9809164.

[TaT2000] D. Tamarkin, B. Tsygan, Non-commutative differential calculus, homotopy BV-algebras and formality conjectures, math.KT/0002116.

[TaT05] D. Tamarkin, B. Tsygan, The ring of differential operators on forms in non-commutative calculus. Proceedings of Symposia in Pure Math., vol. 73, 2005, 105-131.

[ToVa05] B. Toen, M. Vaquie, Moduli of objects in dg-categories, math.AG/0503269.

[Ve77] J. L. Verdier, Categories derivees, etat 0. Lect. Notes in Math. vol 569, 262-312 (1977).

[M] M. Markl, Cotangent cohomology of a category and deformations. J. Pure Appl. Alg., 113:2 (1996), 195-218.

[So97] Y. Soibelman,Meromorphic tensor categories, q-alg/9709030.

[Sw] Switzer, Algebraic topology

[Ve] Verdier